IVP

$$\left.\begin{aligned} \Box u &= 0 \\ u(0,x) &= \phi \\ \tfrac{1}{c}u_t(0,x) &= \psi \end{aligned}\right] \text{ analogous to } \begin{aligned} \ddot{x} &= f(t,x,\dot{x}) \\ x(0) &= x_0 \\ \dot{x}(0) &= y_0 \end{aligned}$$

✓

If there is a solution it has the form

$$f(x-ct) + g(x+ct)$$

$$f(x) + g(x) = \phi(x)$$

$$-f'(x) + g'(x) = \psi(x)$$

$$f' + g' = \phi'$$

$$g' = \phi' + \psi$$

$$f' = \phi' - \psi$$

$$g(x) = \left[\phi(x) + k_1 + \int_0^x \psi(s)\,ds\right]\frac{1}{2}$$

$$f(x) = \left[\phi(x) + k_2 - \int_0^x \psi(s)\,ds\right]1/2$$

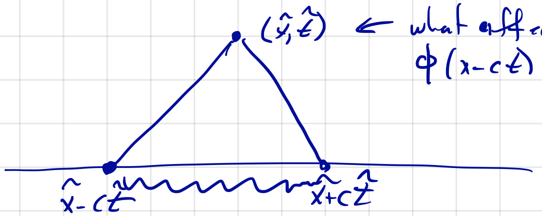
$$f(x) + g(x) = \phi(x) + \underbrace{k_1 + k_2}_{=0}$$

$$k_2 = -k_1$$

$$f(x-ct) + g(x+ct) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2}\int_{x-ct}^{x+ct} \psi(s)\, ds$$

Initial profile splits, $\frac{1}{2} \to$ right, $\frac{1}{2} \to$ left.

Antiderivative of ψ splits $\frac{1}{2} \to (+)$, $\frac{1}{2} \to$ left $(-)$



$\leftarrow$ what affects the value here?

$\phi(x-ct)$, $\phi(x+ct)$ and

$$\int_{x-ct}^{x+ct} \psi(s)\, ds$$

In particular, if ϕ, ψ are 0 between $(\hat{x}-c\hat{t}, \hat{x}+c\hat{t})$ at $t=0$ then $u(\hat{x}, \hat{t}) = 0$.

Logic: If a solution exists, then it's the sum of a left and a right going wave.

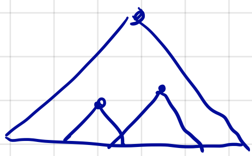
We then used this ansatz to find the only possible sum of a left and right going wave that solves.

(There exists a solution to the IVP and it is unique)

Moreover, we find the principle of causality:

If $\phi, \psi = 0$ on $(x_0 - ct_0, x_0 + ct_0)$ at $t = 0$

then $u(t_0, x_0) = 0$, and indeed throughout



In three dimensions there is more structure.

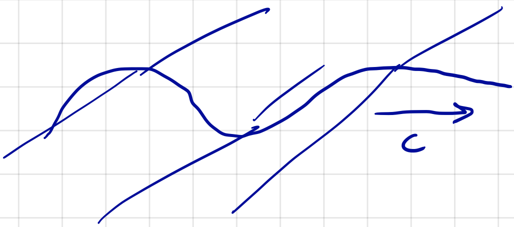
$\xi \in \mathbb{R}^3$ $\quad f(\xi \cdot x - ct) = u(x,t)$

$|\xi| = 1$

$$\frac{1}{c^2} u_{tt} - \Delta u = f''(\xi \cdot x - ct) - f''(\xi \cdot x - ct)\,|\xi|^2$$

$$= 0$$

e.g. $\xi = (1,0,0)$ plane wave travelling right w speed c.



In general this is a plane wave in the ξ direction with speed c.

Among these are the monochromatic waves

$e^{\left(i \frac{\omega}{c}(ct - x\cdot\xi)\right)}$ which have a real imaginary part that each satisfy the wave equation.

$\cos\left(\frac{\omega}{c} x \qquad \right)$ wavelength $\frac{2\pi c}{\omega} = L$

ω: angular freq, $\frac{\text{rad}}{\text{sec}}$.

frequency $\frac{c}{L} = \frac{\omega}{2\pi} = \frac{\text{cycles}}{\text{time}}$

But also, any linear combination of these is again a solution of the wave equation. It's a sum of plane waves being transported in various directions at speed c.

"Energy" density of a solution

$$\frac{1}{2}\left(\frac{1}{c^2}u_t^2 + |\nabla u|^2\right)$$

$$\frac{d}{dt}\int \frac{1}{2}\left(\frac{1}{c^2}u_t^2 + |\nabla u|^2\right)dx$$

$$= \int \frac{1}{c^2}u_t u_{tt} + \nabla u \cdot \nabla u_t \, dx$$

$$= \int \frac{1}{c^2}u_t u_{tt} - u_t \Delta u \, dx$$

$$= \int u_t \left[u_{tt} - \Delta u\right] dx = 0.$$

In fact, the solution has an associated energy-momentum.

$$P = \begin{bmatrix} \frac{1}{2}\left(\frac{|u_t|^2}{c^2} + |\nabla u|^2\right) \\ -\frac{1}{c} u_t \nabla u \end{bmatrix}$$

$$\operatorname{Div} P = \frac{1}{c^3} u_t u_{tt} + \frac{1}{c} \nabla u_t \cdot \nabla u - \frac{1}{c} \nabla u_t \cdot \nabla u - \frac{1}{c} u_t \Delta u$$

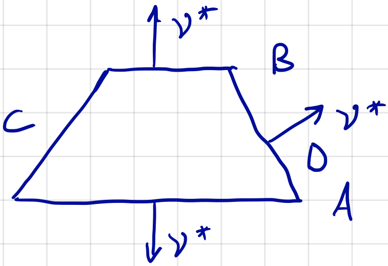
$$= \frac{1}{c} u_t \left[\frac{1}{c^2} u_{tt} - \Delta u \right] = 0.$$

This is a locally conserved quantity.

It is causal, future pointing by Cauchy-Schwartz.

A 2-d computation

$$P = \left[\tfrac{1}{2}\left(\tfrac{1}{c^2}\partial_t u\right)^2 + (\partial_x u)^2 \right]$$



(Euclidean!)

$$0 = -\int_A \tfrac{1}{2}\left[\tfrac{1}{c^2}(\partial_t u)^2 + (\partial_x u)^2\right] + \int_B \tfrac{1}{2}\left[\text{———}\right]$$

$$= \int_C \nu^* \cdot P + \int_D \nu^* \cdot P$$

On D $\quad \nu^* = \frac{1}{\sqrt{2}}[1,1]$

$$\nu^* \cdot P = g(\nu, P) \qquad \nu = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\nwarrow \nu$

On C $\quad \nu^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\nu = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

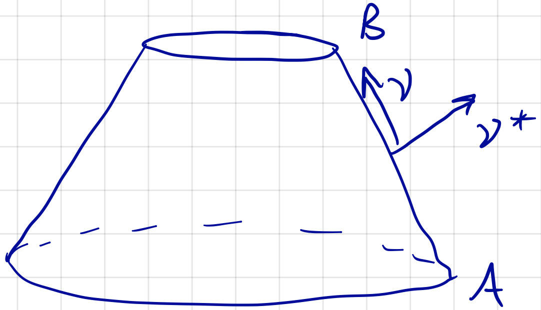
$g(P, \nu) \geqslant 0$ as P is time like, f.p
ν is f.p, null

$$\int_B \frac{1}{2}[\text{——}] = \int_A \frac{1}{2}[\text{——}] - \int_{C \cup D} g(P, \nu)\, dA$$

$$\leqslant \int_A \frac{1}{2}[\text{——}]$$

(with equality iff
P is parallel to ν)

Higher dimens:



$$\int_B \frac{1}{2}\left[\;\text{——}\;\right] = \int_A \frac{1}{2}\left[\;\text{———}\;\right] - \underbrace{\int_C g(\nu, P)}_{\leq 0}$$

If $\partial_t u$, $\nabla u = 0$ on A then also on B

Exercise: u is constant on whole frustum.