Visually, vectors point. They are tangents to curves.

Co-vectors <u>do</u> <u>not</u> <u>point</u>.

$df$ does not point.

But, $\text{Grad} f = (df\, G)^T$ does!

$$df = [\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f]$$

$$\text{Grad} f = \begin{bmatrix} \partial_0 f \\ -\partial_1 f \\ -\partial_2 f \\ -\partial_3 f \end{bmatrix}$$

Moreover
$$\partial_0 f = \partial_t f \cdot \frac{\partial t}{\partial x_0}$$
$$= \frac{1}{c}\partial_t f$$
as $\frac{dx_0}{dt} = c$.

$$df(X) = g(\text{Grad} f, X)$$

Related differential operator: Div

In an inertial coord system let $X(x)$ be a vector field.

$$\mathrm{Div}\, X(x) = \partial_0 X^0 + \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3.$$

If we represent $X$ in a different coord system,

$$\hat{X}(\hat{x}) = L X(\hat{x})$$

$$= L X(Lx)$$

I claim $\quad \widehat{\mathrm{Div}}\, \hat{X}(Lx) = \mathrm{Div}\, X(x)$

$$\hat{\partial}_0 \hat{X}^0 = \partial_i \hat{X}^0 \frac{\partial x_i}{\partial \hat{x}_0} = (L^{-1})_0{}^i \, \partial_i \hat{X}^0$$

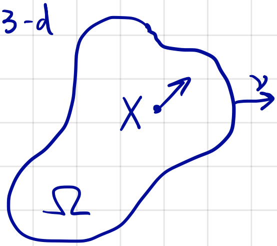
$$= (L^{-1})_0{}^i \, L^0{}_j \, \partial_i X^j$$

$$\sum_{k=0}^{3} \hat{\partial}_k \hat{X}^k = \sum_{k=0}^{3} (L^{-1})_k{}^i \, L^k{}_j \, \partial_i X^j$$

$$= \delta^i{}_j \, \partial_i X^j = \partial_i X^i$$

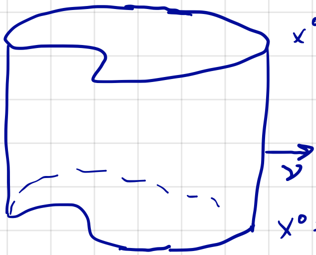
Recall the divergence theorem

2-d, 3-d




$$\int_{\Omega} \operatorname{div} \vec{x} = \int_{\partial\Omega} \vec{x} \cdot \nu \, dV$$

Spacetime




$x^0 = b$

$x^0 = a$

$$X = \begin{bmatrix} x^0 \\ \vec{x} \end{bmatrix}$$

$$\int_{\Omega} \operatorname{div} \vec{x} \, dV = \int_{\partial\Omega} x \cdot \nu$$

$$\int_a^b \int_{\Omega} \operatorname{Div} X \, dU \, dx^0 = \int_a^b \int_{\Omega} \frac{\partial x^0}{\partial x_0} + \operatorname{div} \vec{x} \, dV \, dx_0$$

$$= \int_{\Omega} x^0 \Big|_{x_0 = b} - \int_{\Omega} x^0 \Big|_{x_0 = a} + \int_a^b \int_{\Omega} \operatorname{div} \vec{x} \, dV$$

If we interpret $X^0$ as a density (gunk per volume)

$c\vec{X}$ as a flux gunk per area per time

and $c\vec{X}\cdot\vec{\nu}$ as the rate of flow through the boundary

$$\int X^0\,dV\Big|_{X^0=b} = \int X^0\,dV\Big|_{X^0=a} + \int_a^b \int_{\partial\Omega} c\vec{X}\cdot\vec{\nu}\,dA\,\underbrace{\tfrac{1}{c}\,dx^0}_{dt}$$

$$+ \int_a^b \int_{\Omega} c\,\mathrm{Div}\,X\,dV\,\underbrace{\frac{dx^0}{c}}_{dt}$$

ending amount = starting amount + total flows in/out + production.

$c\,\mathrm{Div}\,X$ gunk / volume / length · length / time

gunk / volume / time

rate of production

Grad takes functions to vector fields
Div takes v.f. to functions

Combine:

$$\hat{\text{Div}}\, L X = \text{Div}\, X$$

$$\hat{\text{Div}}\, \hat{\text{Grad}}\, f = \hat{\text{Div}}\, L\, \text{Grad}\, f$$

$$= \text{Div}\, \text{Grad}\, f$$

$$\Box = \text{Div}\, \text{Grad}$$

$$(\hat{\Box}\hat{f})(Lx) = (\Box f)(x)$$

Compare: $\hat{g}(Lx) = g(x)$

What is $\Box$?

$$\text{Grad}\, f = \begin{bmatrix} \frac{1}{c}\partial_t f \\ -\partial_1 f \\ -\partial_3 f \end{bmatrix}$$

$$\text{Div}\, \text{Grad}\, f = \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t f\right) - \partial_1^2 f - \cdots - \partial_3^2 f$$

$$= \frac{1}{c^2}\partial_t^2 f - \Delta f$$ ⟸ wave operator.

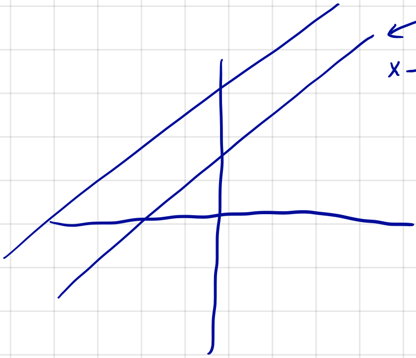
To get a feeling for it, work in 1-d.

$$\frac{1}{c^2} u_{tt} - u_{xx} = 0$$

One solution

$$u = f(\underbrace{x-ct}_{\xi})$$

$u$ constant on these lines



$x - ct = \xi$

This represents a wave traveling right w/ speed $c$.

$u = g(\underbrace{x+ct}_{\eta})$ is also a solution.

And by linearity, so is $f(x-ct) + g(x+ct)$.

I claim every solution has this form

$$\xi = x - ct \qquad x = \frac{\xi + \eta}{2}$$
$$\eta = x + ct$$
$$ct = \frac{\eta - \xi}{2}$$

$$\partial_\xi u = \partial_t u \cdot \frac{\partial t}{\partial \xi} + \partial_x u \frac{\partial x}{\partial \xi}$$

$$= \partial_t u \left(\frac{-1}{c2}\right) + \partial_x u \left(\frac{1}{2}\right)$$

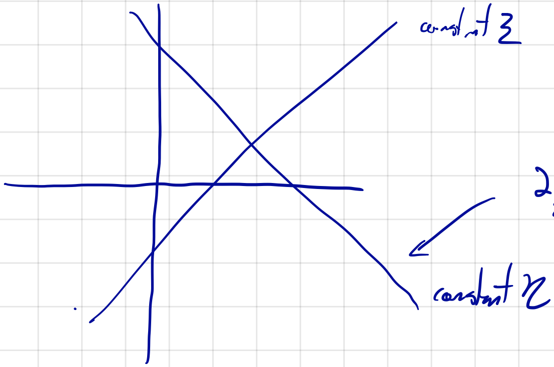
$$= \frac{1}{2}\left[-\frac{1}{c}\partial_t u + \partial_x u\right] \qquad \partial_\xi = \frac{1}{2}\left[\partial_x - \frac{1}{c}\partial_t\right]$$

$$\partial_\eta = \partial_t \cdot \frac{\partial t}{\partial \eta} + \partial_x \frac{\partial x}{\partial \eta}$$
$$= \frac{1}{2}\frac{1}{c}\partial_t + \frac{1}{2}\partial_x$$

$$\partial_\eta \partial_\xi = \frac{1}{4}\left(\frac{1}{c}\partial_t + \partial_x\right)\left(\frac{-1}{c}\partial_t + \partial_x\right)$$

$$= \frac{1}{4}\left[-\frac{1}{c^2}\partial_t^2 + \partial_x^2\right] = -\frac{1}{4}\Box$$

$\partial_\eta \partial_\xi u = 0$ means




$\partial_\xi$ does not change as you

$$\partial_\xi u = g(\xi)$$

$$u = \int g(\xi) + h(\eta)$$

$$= G(\xi) + h(\eta)$$