We will start representing observers in spacetime by timelike vectors of length ~~c~~. 1

What is the energy measured by $\underset{\sim}{U}$?

In a frame where $U = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ it is $cP^0 = c\,g(P, U)$.

In another frame $c\,g(P, U) = c\,g(\hat{P}, \hat{U})$.

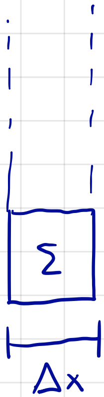
$$cm\gamma(v) \begin{bmatrix} c \\ v \end{bmatrix}$$

So energy is $g(c\underset{\sim}{P}, \underset{\sim}{U})$.

$$cP^0 = c^2 m \gamma(v) \quad \leftarrow \text{observed energy}$$

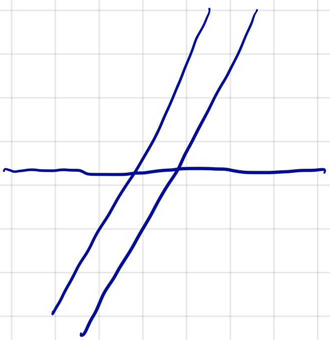
$$c \begin{bmatrix} p^1 \\ p^2 \\ p^3 \end{bmatrix} = \underbrace{c\, m \gamma(v)\, v}_{\text{flux of energy}} = c^2 m \gamma(v) \frac{v}{c}$$

Taking measurements by dot products is common.
Here's another:



particles.

density: $\frac{\Sigma}{\Delta x} = \sigma_0$



Chunk has width $\frac{1}{\gamma}\Delta x$

and new density $\gamma \frac{\Sigma}{\Delta x}$

I.e. New density $= \gamma \sigma_0$

Let me make a vector in the rest frame $\begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix}$

In a frame travelng in which the particles are traveling with velocity v, $\gamma(v) = C$

$$\begin{bmatrix} C & S \\ S & C \end{bmatrix} \begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix} = \begin{bmatrix} C\sigma_0 \\ S\sigma_0 \end{bmatrix}$$

In the time position we have the observed density.

A couple of ways to think about this:

$$\frac{\sigma_0}{c}\, c \underbrace{\begin{bmatrix} C \\ S \end{bmatrix}}_{\text{4-velocity of the particles.}}$$

$$\frac{\sigma_0}{c}\; \underset{\gamma}{C} \begin{bmatrix} c \\ c\,S/c \end{bmatrix} = \frac{\sigma_0}{c}\,\gamma \begin{bmatrix} c \\ v \end{bmatrix}$$

Let's multiply by c

$$\underset{\text{rest density}}{\sigma_0}\; c \underbrace{\begin{bmatrix} C \\ S \end{bmatrix}}_{\text{4-velocity}}$$

components:

$c\sigma_0 C \rightarrow$ c · observed density

$$c\sigma_0 S = c\,\sigma_0 C\left(\frac{S}{C}\right)$$

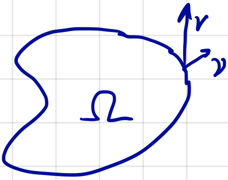
$$= c\sigma_0 C\,\frac{v}{c}$$

$$= \sigma_0 C\, v$$

components are $\begin{bmatrix} c\sigma \\ \sigma v \end{bmatrix}$

observed density · c

particle flux



$$\int_{\partial\Omega} \sigma \; v \cdot \nu \, dA$$

$\frac{\text{number}}{m^3}$ $\frac{m}{s}$ unitless m^2

number/s

number flux

$$c^2 P = \begin{bmatrix} c^3 \gamma(v)\, m \\ m c^2 \gamma(v)\, v \end{bmatrix}$$

c · energy

energy flux

So this is a density-flux vector, N.

If I am observer with 4-velocity V

then in my rest frame

$$g(V, N) = [1, 0]\, G \begin{bmatrix} c\sigma \\ \sigma v \end{bmatrix}$$

$$= c\sigma$$

But this is true in any frame

$$g(\underset{\sim}{V}, \underset{\sim}{N}) = c\sigma$$

Upshot: A distribution of fluid particles is described by a vector at each location. The length of the vector
timelike
encodes $c\cdot$ rest density.

The direction of the vector encodes the (spacetime) 4-velocity; divide by rest density to get the 4-vel of the fluid.

Consider a function on spacetime

$f(t, x)$ in your coordinates.

Give a curve $\alpha(\tau)$ how does the particle see f change.

(Think of f, e.g., as temperature)

$$\frac{d}{d\tau} f(\alpha(\tau)) = \frac{\partial f}{\partial x^0}\frac{d\alpha^0}{d\tau} + \frac{\partial f}{\partial x^1}\frac{d\alpha^1}{d\tau} + \dots + \frac{\partial f}{\partial x^3}\frac{d\alpha^3}{d\tau}$$

$$= \left[\frac{\partial f}{\partial x^0}, \dots, \frac{\partial f}{\partial x^3}\right] \begin{bmatrix} \alpha^0 \\ \vdots \\ \alpha^3 \end{bmatrix}'$$

$$x^0 = ct$$

The four numbers $\left[\frac{\partial f}{\partial x^0}, \dots, \frac{\partial f}{\partial x^3}\right]$ look like they might be the components of a vector. But they are not.

What happens if we change coordinates?

$$\hat{f}(\hat{x}) = f(x(\hat{x}))$$

$$\hat{f}(\hat{x}(x)) = f(x)$$

$$\hat{\alpha} = L\alpha$$

$$\hat{f}(L\alpha) = f(\alpha)$$

$$\left[\frac{\partial \hat{f}}{\partial \hat{x}^0}, \ldots, \frac{\partial \hat{f}}{\partial \hat{x}^3}\right] L\alpha' = \left[\frac{\partial f}{\partial x^0}, \ldots, \frac{\partial f}{\partial x^3}\right]\alpha'$$

This is true for all timelike α' and, as we will see, this implies

$$\left[\frac{\partial \hat{f}}{\partial \hat{x}^0}, \ldots, \frac{\partial \hat{f}}{\partial \hat{x}^3}\right] L = \left[\frac{\partial f}{\partial x^0}, \ldots, \frac{\partial f}{\partial x^3}\right]$$

Compare $\qquad \hat{\alpha}' = L\alpha'$

The L is on the wrong side of the equation, and on the wrong side of the "vector".

The object in question is a co-vector instead.

Representing vectors:

$$\begin{tikzcd} & \underset{\sim}{V} \arrow[dl, "\text{pick coords}"'] \arrow[dr, "\text{pick other coords}"] & \\ V & & \hat{V} \end{tikzcd}$$

column

$\hat{V} = LV$ is the rule for conversion.

Representing covectors

$$\begin{tikzcd} & \underset{\sim}{n} \arrow[dl] \arrow[dr] & \\ n & & \hat{n} \end{tikzcd}$$

row

$$\hat{n} L = n$$

Given a function $\underset{\sim}{f}$ on spacetime, it determines a covector at every point $d\underset{\sim}{f}$. What are its components?

Rep $\underset{\sim}{f}$ in coordinates as f

geometric

$$df = \left[\frac{\partial f}{\partial x^0}, \dots, \frac{\partial f}{\partial x^3} \right]$$

So what is a covector?

Let V be a vector space.

It's dual space V^* is the set of linear maps from V to $\mathbb{R}$. E.g. suppose V is 4-dimensional with basis $e_0, \dots, e_4$.

So any $V \in V$ can be written $v = V^0 e_0 + V^1 e_1 + \dots + V^3 e_3$.

Now let $\eta \in \mathbb{V}^*$. If you know $\eta(e_i) = \eta_i$

then you know $\eta(V)$ for any V!

$$\eta(V) = \eta(V^i e_i) = V^i \eta(e_i) = V^i \eta_i$$

The numbers η_i are the components of η with respect to the basis $e_0, \dots, e_3$.

Concretely:

$\mathbb{V}$ is the set of 4vectors

A covector is just an element of $\mathbb{V}^*$.

$$\begin{array}{ccc} \underset{\sim}{\eta} & \qquad & \underset{\sim}{V} \\ \downarrow & & \downarrow \\ [\eta_0, \dots, \eta_3] & & \begin{bmatrix} V^0 \\ \vdots \\ V^3 \end{bmatrix} \end{array} \qquad \underset{\sim}{\eta}(\underset{\sim}{V}) = \eta_i V^i$$

For this to make sense

$$\hat{\eta}_i \hat{V}^i = \hat{\eta}_i L^i{}_j V^j$$
$$= \eta_j V^j \checkmark$$

Covectors eat vectors and give you numbers, and are linear maps.

We've seen this.

$g(c^2 P, U)$ as a function of U (observed energy)

$\frac{1}{c} g(N, U)$ as a function of U (observed density)

This is the spacetime analog of "take a dot product".

In your past lives you probably confused vectors + covectors because of the structure of dot products.

In fact, there is a way to convert between vectors + covectors in SR as well: we use g/G

I.e. given a vector N define

$$\eta(V) = g(N, V)$$
$$= N^T G V$$

$$N \to \eta = N^T G$$

And given a covector η define a vector

$$N = (\eta G)^T = G^T \eta^T = G \eta^T$$

Note: $N \to \eta \to N$

$$(N^T G G)^T = (N^T)^T = N.$$

Visually, vectors point. They are tangents to curves.

Co-vectors do not point.

df does not point.

But, $\mathrm{Grad}\, f = (df\, G)^T$ does!

$$df = [\partial_0 f, \partial_1 f, \partial_2 f, \partial_3 f]$$

$$\mathrm{Grad}\, f = \begin{bmatrix} \partial_0 f \\ -\partial_1 f \\ -\partial_2 f \\ -\partial_3 f \end{bmatrix}$$

Moreover $\partial_0 f = \partial_t f \cdot \frac{\partial t}{\partial x_0}$

$$= \frac{1}{c} \partial_t f$$

as $\frac{dx_0}{dt} = c.$

$$df(X) = g(\mathrm{Grad}\, f, X)$$

Related differential operator: Div

In an inertial coord system let $X(x)$ be a vector field.

$$\mathrm{Div}\, X(x) = \partial_0 X^0 + \partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3.$$

If we represent X in a different coord system,

$$\hat{X}(\hat{x}) = L X(\hat{x})$$

$$= L X(Lx)$$

I claim $\hat{\mathrm{Div}}\, \hat{X}(Lx) = \mathrm{Div}\, X(x)$

$$\hat{\partial}_0 \hat{X}^0 = \partial_i \hat{X}^0 \frac{\partial x_i}{\partial \hat{x}_0} = (L^{-1})_0{}^i \partial_i \hat{X}^0$$

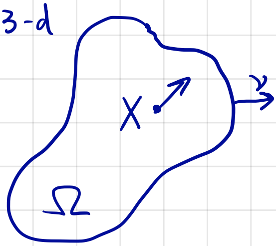
$$= (L^{-1})_0{}^i L^0{}_j \partial_i X^j$$

$$\sum_{k=0}^{3} \hat{\partial}_k \hat{X}^k = \sum_{k=0}^{3} (L^{-1})_k{}^i L^k{}_j \partial_i X^j$$

$$= \delta^i{}_j \partial_i X^j = \partial_i X^i$$

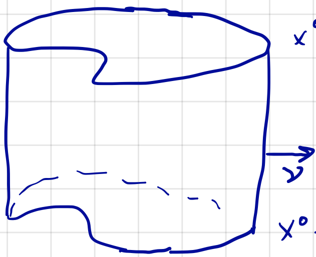
Recall the divergence theorem

2-d, 3-d



$$\int_\Omega \operatorname{div} \vec{x} = \int_{\partial\Omega} \vec{x}\cdot\nu \, dV$$

Spacetime



$x^0 = b$

$x^0 = a$

$$X = \begin{bmatrix} x^0 \\ \vec{x} \end{bmatrix}$$

$$\int_\Omega \operatorname{div} \vec{x} \, dV = \int_{\partial\Omega} x\cdot\nu$$

$$\int_a^b \int_\Omega \operatorname{Div} X \, dV \, dx^0 = \int_a^b \int_\Omega \frac{\partial x^0}{\partial x_0} + \operatorname{div} \vec{x} \, dV \, dx_0$$

$$= \int_\Omega x^0 \Big|_{x_0=b} - \int_\Omega x^0 \Big|_{x_0=a} + \int_a^b \int_\Omega \operatorname{div} \vec{x} \, dV$$

If we interpret X^0 as a density (gunk per volume)

$c\vec{X}$ as a flux gunk per area per time

and $c\vec{X}\cdot\vec{\nu}$ as the rate of flow through the boundary

$$\int X^0\, dV\Big|_{X^0=b} = \int X^0\, dV\Big|_{X^0=a} + \int_a^b \int_{\partial\Omega} c\vec{X}\cdot\vec{\nu}\, dA\, \underbrace{\frac{1}{c}dx^0}_{dt}$$

$$+ \int_a^b \int_{\Omega} c\, \mathrm{Div}\, X\, dV\, \underbrace{\frac{dx^0}{c}}_{dt}$$

ending amount = starting amount + total flows in/out

+ production.

$c\, \mathrm{Div}\, X$ gunk/volume/length · length/time

gunk/volume/time

rate of production