E.g. $\alpha(t) = \begin{bmatrix} ct \\ x(t) \end{bmatrix}$, parameterized by coordinate time.

$$|\alpha'|^2 = g(\alpha',\alpha') = c^2 - |x'|^2$$

$$= c^2 - |v|^2$$

$$= c^2\left(1 - \left|\frac{v}{c}\right|^2\right)$$

$$= c^2\gamma^{-2}$$

$$\frac{c\,\alpha'}{|\alpha'|} = \frac{c\,\alpha'}{c\,\gamma^{-1}} = \gamma\alpha' = \gamma\begin{bmatrix} c \\ x'(t) \end{bmatrix}$$

$$= \gamma\begin{bmatrix} c \\ \vec{v} \end{bmatrix}.$$

Your text uses notation V for $\alpha(s)$'s 4-velocity.

The 4 is old-fashioned.

And although reparameterizing is hard,

computing $\dfrac{d\tau}{ds}$ is easy.

$$\beta(\tau) = \alpha(s(\tau))$$

$$c = |\beta'(\tau)| = |\alpha'(s(\tau))| \frac{ds}{d\tau}$$

$$\boxed{\frac{d\tau}{ds} = \frac{|\alpha'(s)|}{c}}$$

$$\frac{ds}{d\tau} = \frac{c}{|\alpha'|}$$

If the curve is parameterized by coordinate time t

$$\alpha' = \begin{bmatrix} c \\ \vec{v} \end{bmatrix} \quad \text{and} \quad |\alpha'|^2 = c^2\left(1 - \frac{\vec{v}^2}{c^2}\right)$$

$$|\alpha'| = \frac{c}{\gamma(|\vec{v}|)}$$

$$\boxed{\frac{d\tau}{dt} = \gamma^{-1}(\vec{v})}$$

$$\frac{dt}{d\tau} = \gamma(|v|)$$ expresses time dilation.

E.g. Radial motion

$$x(t) = \begin{bmatrix} ct \\ R\cos(\omega t) \\ R\sin(\omega t) \\ 0 \end{bmatrix}$$

$\omega \leq \frac{c}{R}$

for otherwise you travel $2\pi R$ in time $\frac{2\pi}{\omega}$ with speed $R\omega$.

$$\frac{d\tau}{dt} = \frac{1}{c}\left|\frac{dx}{dt}\right| = \frac{1}{c}\sqrt{c^2 - R^2\omega^2}$$

$$= \sqrt{1 - \left(\frac{R\omega}{c}\right)^2}$$

Or: $|v|^2 = (R\omega)^2$ so $\frac{d\tau}{dt} = \gamma(|v|) =$

How much proper time elapses in a single rotation?

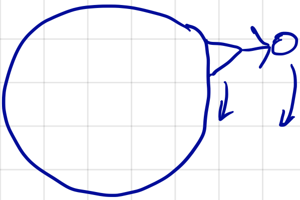
Time for a rotation: $\frac{2\pi}{\omega}$

$$\frac{2\pi}{\omega}\sqrt{1 - \left(\frac{R\omega}{c}\right)^2} \approx \frac{2\pi}{\omega} \text{ if } |R\omega| \ll c.$$

Consequence:

Orbiting at light speed, time does not pass.

Consequence



Head traveling faster.

Your head is younger than your feet

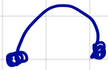
Next HW: how much younger?

Curvature of a curve in the plane

α'' mixes up what the curve is doing with how you parameterize it.

How curvy is it? a) Reparam by arclength

b) $|\alpha''|$ tells you curvature.

e.g. $\alpha(s) = (R\cos(\omega s), R\sin(\omega s))$ 

$$\alpha'' = -\omega^2 (R\cos(\omega s), R\sin(\omega s))$$

$$|\alpha''| = \omega^2 R$$

part for the curve's image

part for the parameterization.

We can eliminate the parameterization dependence by looking at curves parameterized by arclength.

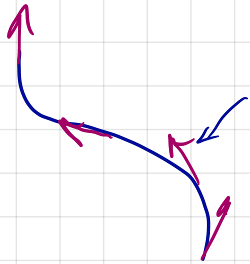
$$\omega = \frac{1}{R}$$

Now $\alpha'' = -\frac{1}{R^2}\alpha$ and $|\alpha''| = \frac{1}{R}$

Tiny circle, huge $1/R$. Big circle, tiny $\frac{1}{R}$.

We call this quantity $|\alpha''|$ when α is paramed by arclength the curvature of the curve.

Units: $1/L$.



always unit length.

How can α' change?

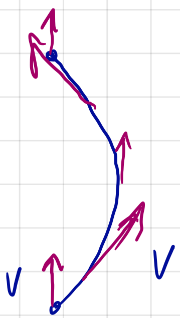
$$\alpha' \cdot \alpha' = 1$$

$$\text{so } \frac{d}{ds} \alpha' \cdot \alpha' = 0$$

$$= 2\alpha' \cdot \alpha''$$

So α'' is always perp to α'. All α' can do is rotate.

Space time picture



$$|V| = c$$

Analogously, 4-velocity: $\dfrac{d\alpha}{d\tau} = V$

4-acceleration: $\dfrac{d^2\alpha}{d\tau^2}$

$$V(s) = c\,\frac{\alpha'}{|\alpha'|} \qquad \frac{d\tau}{ds} = \frac{1}{c}\,|\alpha'|$$

$$\frac{dV}{d\tau} = \frac{dV}{ds}\cdot\frac{ds}{d\tau} = \underbrace{\frac{dV}{ds}}_{\text{Typically a mess.}}\cdot\frac{c}{|\alpha'|}$$

In fact $\quad V(s) = \dfrac{c\,\alpha'}{(|\alpha'|^2)^{1/2}}$

$$\frac{dV}{ds} = \frac{c\,\alpha''}{|\alpha'|} + c\alpha'\left(\frac{-1}{2}\right)\frac{1}{|\alpha'|^3}\,2\,\alpha'\cdot\alpha''$$

$$= \frac{c\,\alpha''}{|\alpha'|} - c\,\frac{(\alpha'\cdot\alpha'')}{|\alpha'|^3}\,\alpha'$$

$$= \frac{c}{|\alpha'|}\left[\,|\alpha'|^2\alpha'' - (\alpha'\cdot\alpha'')\alpha'\,\right]$$

$$\frac{dV}{d\tau} = \left(\frac{c^2}{|\alpha'|^4}\right)\left[\,|\alpha'|^2\alpha'' - \alpha'\cdot\alpha''\alpha'\,\right]$$

e.g. curves parameterized by coord. time

$$\alpha' = \begin{bmatrix} c \\ \vec{v} \end{bmatrix}$$

$$|\alpha'| = c\gamma(|v|)^{-1}$$

$$\frac{dV}{d\tau} = \frac{\gamma^4}{c^2}\left[c^2\gamma^{-2}\alpha'' - g(\alpha',\alpha'')\alpha'\right]$$

$$= \gamma^2 \begin{bmatrix} 0 \\ v' \end{bmatrix} - \frac{\gamma^4}{c^2} g\left[\begin{bmatrix} c \\ \vec{v} \end{bmatrix}, \begin{bmatrix} 0 \\ \vec{v}' \end{bmatrix}\right]\alpha'$$

$$= \gamma^2 \begin{bmatrix} 0 \\ v' \end{bmatrix} + \frac{\gamma^4}{c^2}(\vec{v}\cdot\vec{v}')\begin{bmatrix} c \\ \vec{v} \end{bmatrix}$$

Exercise: $|\vec{v}| \frac{d}{dt}|\vec{v}| = \vec{v}\cdot\vec{v}'$ to get text's description

$$\frac{dV}{d\tau} = A, \qquad g(A,A) = a^2$$

In a frame where $\vec{v} = 0$

$$A = \begin{bmatrix} 0 \\ v' \end{bmatrix}, \qquad a = |v'|.$$

e.g. radial motion

$$\alpha(t) = \begin{bmatrix} ct \\ R\cos(\omega t) \\ R\sin(\omega t) \\ 0 \end{bmatrix} \qquad |R\omega| < c$$

$$\alpha' = \begin{bmatrix} c \\ -R\omega \sin(\omega t) \\ R\omega\cos(\omega t) \\ 0 \end{bmatrix}$$

$$|\alpha'|^2 = c^2 - R^2\omega^2$$

$$= c^2\left(1 - \left(\frac{R\omega}{c}\right)^2\right)$$

$$V = \frac{1}{\left(1-\left(\frac{R\omega}{c}\right)^2\right)^{1/2}} \begin{bmatrix} c \\ -R\omega\sin(\omega t) \\ R\omega\cos(\omega t) \end{bmatrix}$$

$$\frac{dt}{d\tau} = \frac{c}{|\alpha'|} \quad (\text{unitless!}) \quad = \frac{1}{\left(1-\left(\frac{R\omega}{c}\right)^2\right)^{1/2}}$$

$$A = \frac{dV}{d\tau} = \frac{dV}{dt}\frac{dt}{d\tau} = \frac{-\omega^2}{\left(1-\left(\frac{R\omega}{c}\right)^2\right)} \begin{bmatrix} 0 \\ R\cos(\omega t) \\ R\sin(\omega t) \end{bmatrix}$$

In the plane

$$\alpha'(s) = \begin{bmatrix} \cos(\theta(s)) \\ \sin(\theta(s)) \end{bmatrix}$$ for a unit speed curve.

$$\alpha''(s) = \underbrace{\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}}_{(\alpha')^{\perp}} \theta'(s)$$

$$|\alpha''(s)| = |\theta'(s)|$$

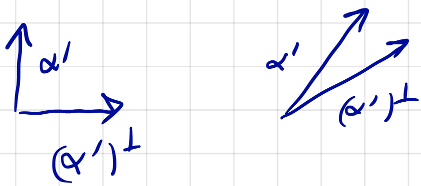
and $\theta' > 0 \Rightarrow$ turn left
$ < 0 \Rightarrow$ turning right

$|\theta'(s)|$ is the curvature of the curve.

Ditto: in $1+1$ Lorenzian case

$$\alpha'(\tau) = c\begin{bmatrix} \cosh(\psi(\tau)) \\ \sinh(\psi(\tau)) \end{bmatrix} \quad \text{for some } \tau.$$

$$\alpha''(\tau) = \underbrace{c\begin{bmatrix} \sinh(\psi(\tau)) \\ \cosh(\psi(\tau)) \end{bmatrix}}_{(\alpha')^{\perp}} \psi'(\tau)$$



$$|\alpha''(\tau)| = c\,|\psi'(\tau)|$$

I'll still call $\frac{1}{c}|\alpha''|$ the curvature of the curve.

What does constant acceleration look like?

Plane $\quad |\alpha''| = \kappa$

$\hookrightarrow |\theta'| = |\alpha''|$ so $\theta' \equiv \kappa$ or $\theta' \equiv -\kappa$

$$\theta = \kappa s + s_0$$

$$\alpha' = \begin{bmatrix} -\sin(\kappa s + s_0) \\ \cos(\kappa s + s_0) \end{bmatrix}$$

$$\alpha = \frac{1}{\kappa} \begin{bmatrix} \cos(\kappa s + s_0) \\ \sin(\kappa s + s_0) \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$\hookrightarrow$ traverse a circle of radius $\frac{1}{\kappa}$.

Spacetime:

$$\alpha''(\tau) = c \begin{bmatrix} \sinh(\psi(\tau)) \\ \cosh(\psi(\tau)) \end{bmatrix} \psi'(\tau)$$

$$|\alpha''| = c \underbrace{|\psi'(\tau)|}_{K}$$

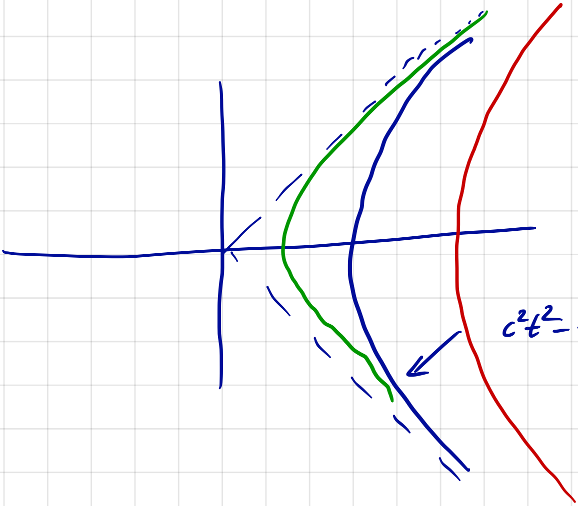
$$\psi(\tau) = K\tau + \psi_0$$

$$\alpha'(\tau) = c \begin{bmatrix} \cosh(K\tau + \psi_0) \\ \sinh(K\tau + \psi_0) \end{bmatrix}$$

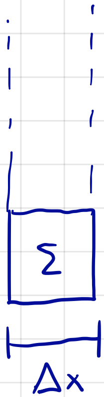
$$\alpha(\tau) = \frac{c}{K} \begin{bmatrix} \sinh(K\tau + \psi_0) \\ \cosh(K\tau + \psi_0) \end{bmatrix} + \underbrace{\begin{bmatrix} ct_0 \\ x_0 \end{bmatrix}}_{0 \text{ WLOG}}$$

$$c^2t^2 - x^2 = \frac{c^2}{K^2}\left[S^2 - C^2 \right]$$

$$= -\frac{c^2}{K^2}$$

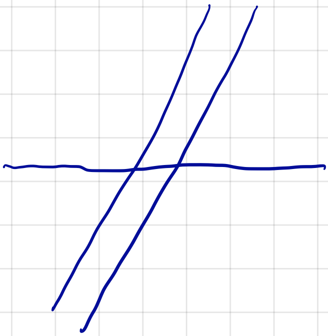


$$c^2t^2 - x^2 = -\frac{c^2}{K^2}$$



particles.

density: $\frac{\Sigma}{\Delta x} = \sigma_0$



Chunk has width $\frac{1}{\gamma}\Delta x$

and new density $\gamma \frac{\Sigma}{\Delta x}$

I.e. New density $= \gamma \sigma_0$

Let me make a vector in the rest frame $\begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix}$

In a frame traveling in which the particles are traveling with velocity v, $\gamma(v) = C$

$$\begin{bmatrix} C & S \\ S & C \end{bmatrix} \begin{bmatrix} \sigma_0 \\ 0 \end{bmatrix} = \begin{bmatrix} C\sigma_0 \\ S\sigma_0 \end{bmatrix}$$

In the time position we have the observed density.

A couple of ways to think about this:

$$\frac{\sigma_0}{c}\, \underbrace{c \begin{bmatrix} C \\ S \end{bmatrix}}_{\text{4-velocity of the particles.}}$$

Let's multiply by c

$$\underbrace{\sigma_0}_{\text{rest density}}\, \underbrace{c \begin{bmatrix} C \\ S \end{bmatrix}}_{\text{4-velocity}}$$

components: $c\sigma_0 C \rightarrow$ $c \cdot$ observed density

$$c\sigma_0 S = c\,\sigma_0 C\left(\frac{S}{C}\right)$$

$$= c\sigma_0 C\,\frac{v}{c}$$

$$= \sigma_0 C\, v$$

components are $\begin{bmatrix} c\sigma \\ \sigma v \end{bmatrix}$

observed density

particle flux

So this is a density-flux vector, N.

If I am observer with 4-velocity V

then in my rest frame

$$g(V, N) = [c, 0]\, G \begin{bmatrix} c\sigma \\ \sigma v \end{bmatrix}$$

$$= c^2 \sigma$$

But this is true in any frame

$$g(\underset{\sim}{V}, \underset{\sim}{N}) = c^2 \sigma$$