Last class:

$\alpha(s)$: curve in spacetime

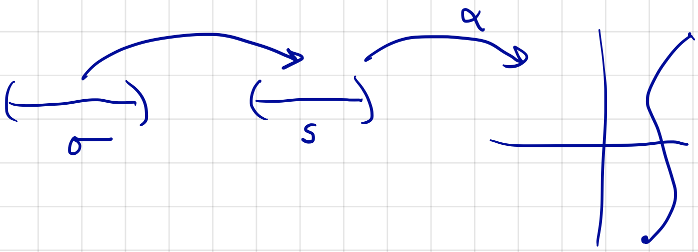
$$\alpha(s) = \begin{bmatrix} ct(s) \\ \vec{x}(s) \end{bmatrix}$$

$\alpha'(s)$ transforms as a vector. A curve is <u>causal</u> if α' always is.

$$\int_a^b \frac{1}{c}\sqrt{g(\alpha', \alpha')}\, ds = \Delta\tau,$$ change in proper time along curve.

elapsed time for the traveller, from $s = a$ to $s = b$.

A curve can be reparameterized:



$\alpha(s(\sigma))$ We require $\frac{ds}{d\sigma} \geq 0$ so that there is no double backing.

between points along a causal curve

Theorem: Proper time is independent of the parametrization

Pf: Let $\alpha(s)$ be a causal curve and consider the proper time between $\alpha(s_0)$ and $\alpha(s_1)$.

Let

$$\beta(\sigma) = \alpha(s(\sigma)).$$

be a reparameterization with $s_0 = s(\sigma_0)$, $s_1 = s(\sigma_1)$.

Then $\beta'(\sigma) = \alpha'(s(\sigma))\dfrac{ds}{d\sigma}$ and

$$\sqrt{g(\beta',\beta')} = \sqrt{g(\alpha',\alpha')}\,\frac{ds}{d\sigma}, \text{ using } \frac{ds}{d\sigma} \geq 0.$$

By change of vars,

$$\int_{\sigma_0}^{\sigma_1} \frac{1}{c}\sqrt{g(\beta',\beta')}\,d\sigma = \int_{\sigma_0}^{\sigma_1} \frac{1}{c}\sqrt{g(\alpha',\alpha')}\,\frac{ds}{d\sigma}\,d\sigma$$

$$= \int_{s_0}^{s_1} \frac{1}{c}\sqrt{g(\alpha',\alpha')}\,ds.$$

Def: A causal curve $\alpha(s)$ is <u>parameterized by proper time</u> if $g(\alpha', \alpha') = c^2$ along the curve.

Note, for such a curve,

$$\int_a^b \frac{1}{c}\sqrt{g(\alpha',\alpha')}\, ds = \int_a^b 1\, ds = b-a.$$

How much proper time elapses from $s=a$ to $s=b$? $b-a$.

The parameter encodes this.

If α is a curve in one coord system and $\begin{bmatrix}\hat{t}\\ \hat{x}\end{bmatrix} = C^{-1}LC\begin{bmatrix}t\\ x\end{bmatrix} + Z$,

then $\hat{\alpha} = L\alpha + Z$. (The C's are built in).

So $\hat{\alpha}' = L\alpha'$. In particular,

$$g(\hat{\alpha}', \hat{\alpha}') = g(L\alpha', L\alpha') = g(\alpha', \alpha').$$

So if α is parameterized by arclength so is $\hat{\alpha}$.

A geometric analogy:

Def: A curve $\alpha(s)$ in the plane $(\mathbb{R}^2)$ or $\mathbb{R}^3$ is parameterized by arclength if $|\alpha'(s)|=1$ for all s.

e.g. $\alpha(s) = [R\cos(s/R), R\sin(s/R)]$ is parameterized by arclength. $\alpha'(s) = [\cos(s/R), \sin(s/R)]$.

How long to go around the circle? $2\pi R$.

How far around the circle? $2\pi R$.

As s goes up by 1, so does arclength

Given any curve in the plane with $\alpha' \neq 0$

we can reparameterize it by arclength.

$$\beta(\sigma) = \alpha(s(\sigma))$$

$$\beta'(\sigma) = \alpha'(s(\sigma))\, s'(\sigma)$$

$$1 = |\alpha'(s(\sigma))| \frac{ds}{d\sigma}$$

So
$$\frac{ds}{d\sigma} = \frac{1}{|\alpha'(s\quad)|} \qquad (\text{uses } |\alpha'(s)| \neq 0)$$

which is an ODE to solve for s. In fact

$$|\alpha'(s)|\, ds = d\sigma$$

So

$$\sigma = \int_{s_0}^{s} |\alpha'(t)|\, dt.$$

Now solve for $s(\sigma)$. → possible, in principle: $\frac{d\sigma}{ds} > 0$.

But in practice, this is awful.

1) Solve an impossible integral

2) Solve an impossible algebraic eq.

Ditto, given a timelike curve, we can always parameterize it by proper time.

Now, given any curve $\alpha(s)$, we can determine its tangent vector at any point if it were parameterized by proper time. It's just

$$c \frac{\alpha'(s)}{\sqrt{g(\alpha', \alpha')}}$$ which has length c.

This rescaling is the infinitesimal version of reparameterizing

Def The 4-velocity of a timelike curve $\alpha(s)$ is

$$c \frac{\alpha'(s)}{\sqrt{g(\alpha', \alpha')}}$$

It's just the velocity of a reparameterized curve.

When α' is timelike, I'll write $|\alpha'|$ rather than $\sqrt{g(\alpha', \alpha')}$

E.g. $\alpha(t) = \begin{bmatrix} ct \\ x(t) \end{bmatrix}$, parameterized by coordinate time.

$$|\alpha'|^2 = g(\alpha', \alpha') = c^2 - |\cancel{x}'|^2$$

$$= c^2 - |v|^2$$

$$= c^2 \left(1 - \left|\frac{v}{c}\right|^2\right)$$

$$= c^2 \gamma^{-2}$$

$$\frac{c\alpha'}{|\alpha'|} = \frac{c\alpha'}{c\gamma^{-1}} = \gamma\alpha' = \gamma \begin{bmatrix} c \\ x'(t) \end{bmatrix}$$

$$= \gamma \begin{bmatrix} c \\ \vec{V} \end{bmatrix}.$$

Your text uses notation V for $\alpha(s)$'s 4-velocity.

The 4 is old-fashioned.

And although reparameterizing is hard, computing $\frac{d\tau}{ds}$ is easy.

$$\beta(\tau) = \alpha(s(\tau))$$

$$c = |\beta'(\tau)| = |\alpha'(s(\tau))| \frac{ds}{d\tau}$$

$$\boxed{\frac{d\tau}{ds} = \frac{|\alpha'(s)|}{c}} \qquad \frac{ds}{d\tau} = \frac{c}{|\alpha'|}$$

If the curve is parameterized by coordinate time t

$$\alpha' = \begin{bmatrix} c \\ \vec{v} \end{bmatrix} \quad \text{and} \quad |\alpha'|^2 = c^2\left(1 - \frac{\vec{v}^2}{c^2}\right)$$

$$|\alpha'| = \frac{c}{\gamma(|\vec{v}|)}$$

$$\boxed{\frac{d\tau}{dt} = \gamma^{-1}(\vec{v})} \qquad \frac{dt}{d\tau} = \gamma(|v|) \text{ expresses time dilation.}$$

E.g. Radial motion

$$\alpha(t) = \begin{bmatrix} ct \\ R\cos(\omega t) \\ R\sin(\omega t) \\ 0 \end{bmatrix}$$

$\omega \leq \frac{c}{R}$

for otherwise you travel $2\pi R$

in time $\frac{2\pi}{\omega}$ with speed $R\omega$.

$$\frac{d\tau}{dt} = \frac{1}{c}\left|\frac{d\alpha}{dt}\right| = \frac{1}{c}\sqrt{c^2 - R^2\omega^2}$$

$$= \sqrt{1 - \left(\frac{R\omega}{c}\right)^2}$$

Or: $|v|^2 = (R\omega)^2$ so $\frac{d\tau}{dt} = \gamma(|v|) =$

How much proper time elapses in a single rotation?

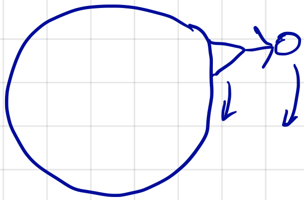
Time for a rotation: $\frac{2\pi}{\omega}$

$$\frac{2\pi}{\omega}\sqrt{1 - \left(\frac{R\omega}{c}\right)^2} \approx \frac{2\pi}{\omega} \text{ if } |R\omega| \ll c.$$

Consequence:

Orbiting at light speed, time does not pass.

Consequence



Head traveling faster.

Your head is younger than your feet

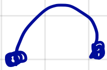
Next HW: how much younger?

Curvature of a curve in the plane

α'' mixes up what the curve is doing with how you parameterize it.

How curvy is it? a) Reparam by arclength

b) $|\alpha''|$ tells you curvature.

e.g. $\alpha(s) = \left(R\cos(\omega s), R\sin(\omega s)\right)$ 

$$\alpha'' = -\omega^2\left(R\cos(\omega s), R\sin(\omega s)\right)$$

$$|\alpha''| = \omega^2 R$$

part for the curve's image

part for the parameterization.

We can eliminate the parameterization dependence by looking at curves parameterized by arclength.

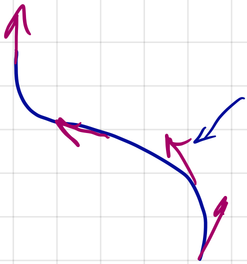
$$\omega = \frac{1}{R}$$

Now $\alpha'' = -\frac{1}{R^2}\alpha$ and $|\alpha''| = \frac{1}{R}$

Tiny circle, huge $1/R$. Big circle, tiny $\frac{1}{R}$.

We call this quantity $|\alpha''|$ when α is paramed by arclength the curvature of the curve.

Units: $1/L$.



always unit length.

How can α' change?

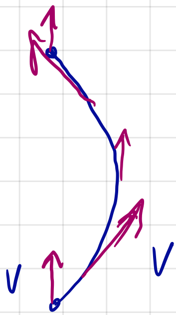
$$\alpha' \cdot \alpha' = 1$$

$$\text{so } \frac{d}{ds} \alpha' \cdot \alpha' = 0$$

$$\qquad = 2\alpha' \cdot \alpha''$$

So α'' is always perp to α'. All α' can do is rotate.

Spruce the picture



$$|V| = c$$

Analogously,

4-velocity: $\frac{d\alpha}{d\tau} = V$

4-acceleration: $\frac{d^2\alpha}{d\tau^2}$

$$V(t) = c\frac{\alpha'}{|\alpha'|} \qquad \frac{d\tau}{dt} = \frac{1}{c}|\alpha'|$$

$$\frac{dV}{d\tau} = \frac{dV}{dt}\cdot\underbrace{\frac{dt}{d\tau}}_{\gamma(|v|)} = \underbrace{\frac{dV}{dt}}_{\text{Typically a mess}}\cdot\frac{c}{|\alpha'|}$$

$$\alpha = \begin{bmatrix} ct \\ R\cos(\omega t) \\ R\sin(\omega t) \end{bmatrix} \qquad |\alpha'| = \sqrt{c^2 - R^2\omega^2} = c\sqrt{1-\left(\frac{R\omega}{c}\right)^2}$$

$$V = \frac{1}{\sqrt{1-\left(\frac{R\omega}{c}\right)}}\begin{bmatrix} c \\ -R\omega\sin \\ R\omega\cos \end{bmatrix}$$

$$\frac{dU}{d\tau} = \frac{1}{\sqrt{1-\left(\frac{R\omega}{c}\right)^2}}\begin{bmatrix} 0 \\ -R\omega^2\cos(\omega t) \\ -R\omega^2\sin(\omega t) \end{bmatrix}\frac{1}{\sqrt{1-\left(\frac{R\omega}{c}\right)^2}}$$

$$= \frac{1}{\left(1-\left(\frac{R\omega}{c}\right)^2\right)} \begin{bmatrix} 0 \\ -\omega^2 R\cos(\omega t) \\ -\omega^2 R \sin(\omega t) \end{bmatrix}$$

usual acceleration

Newtonian

relativistic correction factor

Is infinite if $R\omega = c$!

e.g. $\alpha(t) = \begin{bmatrix} ct \\ x(t) \end{bmatrix}$

$$\frac{d\alpha}{d\tau} = \gamma(v) \begin{bmatrix} c \\ v \end{bmatrix}$$

$$\frac{d\tau}{dt} = \frac{1}{c}\sqrt{c^2 - |v|^2}$$

$$= \gamma^{-1}$$

$$\frac{d^2\alpha}{d\tau^2} = \frac{dt}{d\tau}\frac{d}{dt}\left[\gamma \begin{bmatrix} c \\ v \end{bmatrix}\right]$$

$$dt = \gamma\, d\tau$$

$$= \gamma \left[\gamma' \begin{bmatrix} c \\ v \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ v' \end{bmatrix}\right]$$

$$= (\gamma\gamma') \begin{bmatrix} c \\ v \end{bmatrix} + \gamma^2 \begin{bmatrix} 0 \\ v' \end{bmatrix}$$

$$\frac{d}{dt} \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{-1}{2} \frac{1}{\left(1-\frac{v}{c}\right)^{3/2}} (-2) \frac{v}{c} \frac{v'}{c}$$

$$= \gamma^3 \frac{v}{c^2} \frac{dv}{dt}$$

$$\gamma \gamma' = \gamma^4 \frac{|v|}{c^2} \frac{d}{dt} |v|$$