Last class: introduced the notion of a parameterized curve in space time.

Suppose we have a particle (observer, planet, etc.)
electron

and you wish to describe its motion in your inertial coord system.

At each time t you know the location $\vec{x}(t)$ of the particle. From this you obtain a curve

$$\alpha(t) = \begin{bmatrix} t \\ \vec{x}(t) \end{bmatrix}$$

Some other observer could do the same thing

$$\hat{\alpha}(\hat{t}) = \begin{bmatrix} \hat{t} \\ \vec{\hat{x}}(\hat{t}) \end{bmatrix}.$$

And you could convert:

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \gamma \begin{bmatrix} C & S \\ S & C \end{bmatrix} \begin{bmatrix} c\hat{t} \\ \hat{x}(\hat{t}) \end{bmatrix}$$

$$ct = \gamma\left(C c\hat{t} + S\hat{x}(\hat{t})\right)$$

$$x = \gamma\left(S c\hat{t} + C\hat{x}(\hat{t})\right)$$

I.e $\alpha(t) = \begin{bmatrix} t \\ x(t) \end{bmatrix}$

$$\beta(\hat{t}) = \begin{bmatrix} \gamma C\hat{t} + \gamma/c\, S\hat{x}(\hat{t}) \\ \gamma S c\hat{t} + C\hat{x}(\hat{t}) \end{bmatrix}$$

are two equally valid descriptions of the same particle in your coordinate system.

The only thing that is different is the parameter t vs $\hat{t}$ used to describe the curve

So we'll be agnostic

$$\alpha(s) = \begin{bmatrix} t(s) \\ \vec{x}(s) \end{bmatrix}$$

$$\alpha'(s) = \begin{bmatrix} t' \\ \vec{x}' \end{bmatrix}$$

Your observed velocity: $v = \dfrac{d\vec{x}/ds}{dt/ds}$

$$\mathcal{C}\alpha(s) = \begin{bmatrix} ct(s) \\ \vec{x}(s) \end{bmatrix}$$

in another system $\hat{\alpha} = \mathcal{C}^{-1} L \mathcal{C} \alpha + Z$

$$\hat{\alpha}' = \mathcal{C}^{-1} L \mathcal{C} \alpha'$$

$$\mathcal{C}\hat{\alpha}' = L \mathcal{C}\alpha'$$

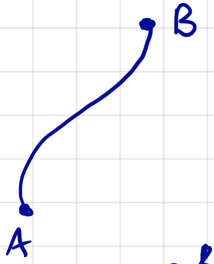
So $\mathcal{C}\alpha'$ transforms as a vector. For this reason, we will henceforth write our curves α in geometric coordinates $\alpha(s) = \begin{bmatrix} ct(s) \\ \vec{x}(s) \end{bmatrix}$

instead of above.

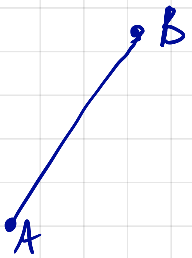
I.e. $\alpha'(s)$ is a vector for each s. Causal if observed velocity $\le c$.

$$\frac{|d\vec{x}/ds|}{|dt/ds|} \le c \iff \left|d\vec{x}/ds\right|^2 \le c^2 \frac{dt^2}{ds^2}$$

$$\iff c^2\left(\frac{dt}{ds}\right)^2 - \left|\frac{d\vec{x}}{ds}\right|^2 \ge 0$$



How much time elapses for the particle?

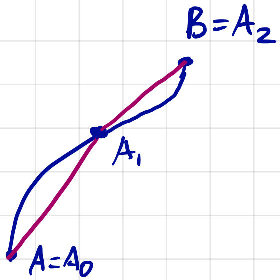


If non accelerating,

$$X = C(B-A) = \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix}$$

$$\tau = \frac{1}{c}\sqrt{X^T G X} = \frac{1}{c}\sqrt{g(X,X)}$$

square root of interval, converted to time.



$$\Delta\tau_1 = \frac{1}{c}\sqrt{g(X_1, X_1)}$$

$$X_i = e(A_i - A_{i-1})$$

$$\Delta\tau_2 = \frac{1}{c}\sqrt{g(X_2, X_2)}$$

$$\Delta\tau = \Delta\tau_1 + \Delta\tau_2$$

For finitely many:

$E_k = \alpha(s_k)$ (already in geometric coords)

$$X_k = E_k - E_{k-1}$$

$$\Delta\tau = \sum_{k=1}^{n} \frac{1}{c}\sqrt{g(X_k, X_k)}$$

But, if the steps are really small

$$E_{k+1} = \alpha(s_{k+1}) \approx \alpha(s_k) + \alpha'(s_k)\underbrace{(s_{k+1} - s_k)}_{\Delta s_{k+1}}$$

$$= E_k + \alpha'(s_k)\Delta s_{k+1}$$

$$X_{k+1} \approx \alpha'(s_k)\Delta s_{k+1}$$

$$\frac{1}{c}\sqrt{g(X_k, X_k)} \approx \frac{1}{c}\sqrt{g(\alpha'(s_{k-1}), \alpha'(s_{k-1}))}\,\Delta s_k$$

In the limit

$$\Delta \tau = \int_{s_0}^{s_1} \frac{1}{c}\sqrt{g(\alpha'(s), \alpha'(s))}\, ds$$

Proper time along the curve.

If we set $\tau = 0$ at s_0

$$\tau = \int_{s_0}^{s} \frac{1}{c}\sqrt{g(\alpha'(\sigma), \alpha'(\sigma))}\, d\sigma$$

$$\frac{d\tau}{ds} = \frac{1}{c}\sqrt{g(\alpha'(\sigma), \alpha'(\sigma))}$$

conversion factor from parameter s to proper time.