Now 3 space dimensions.

We'll assume coordinate transformations are affine (so non-acceleration is preserved).

We'll assume that if one observer says $E$ and $F$ lie on the path of a non accelerating particle traveling at the speed of light, all observers say so.

$$E = (t_0, x_0) \qquad \rightarrow \qquad |c\Delta t| = |\Delta x|$$
$$F = (t_1, x_1) \qquad\qquad |c\Delta t'| = |\Delta x'|$$

Let us deal with linear case first

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = A \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\downarrow$$

$$C^{-1} L C$$

$$C = \begin{bmatrix} c & \\ & 1 \end{bmatrix}$$

$F$ lies on path of a photon from $O$:

$$c\,|t| = |x|$$

$$c^2 (t)^2 = |x|^2$$

$$(\mathcal{C}F)^T G\, \mathcal{C}F = 0 \qquad G = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

We require

$$(\mathcal{C}AF)^T G\, \mathcal{C}AF = 0$$

whenever $(\mathcal{C}F)^T G\, \mathcal{C}F = 0$

$$A = \mathcal{C}^{-1} L\, \mathcal{C} \qquad (\mathcal{C}F)^T L^T G\, L\, \mathcal{C}F = 0$$

whenever

I.e. $\quad X^T L^T G\, L\, X = 0$ whenever

$$X^T G X = 0 \qquad (X = \mathcal{C}F)$$

e.g: $X = \begin{bmatrix} 1 \\ U \end{bmatrix}$ $\quad |U|^2 = 1$

$L^T G L = \begin{bmatrix} \alpha & a \\ a & S \end{bmatrix}$ for some $\alpha, a, S$ $\quad S^T = S$

(why?)

$$X^T L^T G L X = \alpha + 2a \cdot U + U^T S U$$

$$= U^T(\alpha I)U + 2a \cdot U + U^T S U$$

$$= U^T[\alpha I + S]U + 2a U$$

So $\quad U^T[\alpha I + S]U + 2a \cdot U = 0 \quad$ if $|U| = 1$

$U \to -U$

$$U^T[\alpha I + S]U - 2a \cdot U = 0 \quad \text{also } ①$$

So $a \cdot U = 0$ for all unitvectors $U$,

and therefore all vectors. So $a \cdot a = 0$ and $a = 0$.

Also:

$U^T[\alpha I + S]U = 0$ for all unit vectors $U$.

But any $W = \lambda U$, $U$ a unit vector, so

$$W^T[\alpha I + S]W = \lambda^2 U^T[\alpha I + S]U = 0$$

for all vectors $W \in \mathbb{R}^3$.

$B(W, Y) = W^T[\alpha I + S]Y$ is bilinear and

agrees on diagonal with $0$. So $W^T[\alpha I + S]Y = 0$

for all vectors $W, Y$. So $\alpha I + S = 0$ (use $e_i, e_j$).

So $S = -\alpha I$

We conclude: $a = 0 \quad ( a \cdot a = 0 ! )$

$$S + \alpha I = 0, \quad S = -\alpha I \quad \text{(polarization)}$$

$$L^T G L = \alpha \begin{bmatrix} 1 & \\ & -I \end{bmatrix} = \alpha G.$$

Now $$L = \begin{bmatrix} \gamma & \text{---} \\ | & * \end{bmatrix}$$

$$L \begin{bmatrix} ct \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma \\ | \end{bmatrix} ct$$

path of $O$.

$$ct' = \gamma ct$$

encodes time dilation.

Relativity hypothesis: time dilation between the two coordinate systems is the same.

$$L^{-1} = \begin{bmatrix} \gamma & \text{---} \\ | & * \end{bmatrix}$$ also, for other stuff.

$L^T G L = \alpha G \Rightarrow$

$L^T = \alpha G L^{-1} G$ $\qquad$ $G \begin{bmatrix} \gamma \\ * \end{bmatrix} = \begin{bmatrix} \gamma \\ -* \end{bmatrix}$

upper left entry is $\gamma$

upper left entry is $\alpha\gamma$

So $\alpha = 1$.

We assumed:

1) Linear
2) path of photons preserved
3) relativity of time dilation

$\begin{bmatrix} t' \\ x' \end{bmatrix} = A \begin{bmatrix} t \\ x \end{bmatrix}$ $\qquad$ $A = C^{-1} L C$ $\qquad$ $L^T G L = G$

These are the (natural) Lorentz transformations.

physical Lorentz transf.

$O(1,3)$: subgroup of $GL(4,\mathbb{R})$ $\quad L^T G L = G.$

Exercise:
The affine transformations are

$$C^{-1} L C + Z \qquad Z \in \mathbb{R}^4$$

$$L \in O(1,3)$$

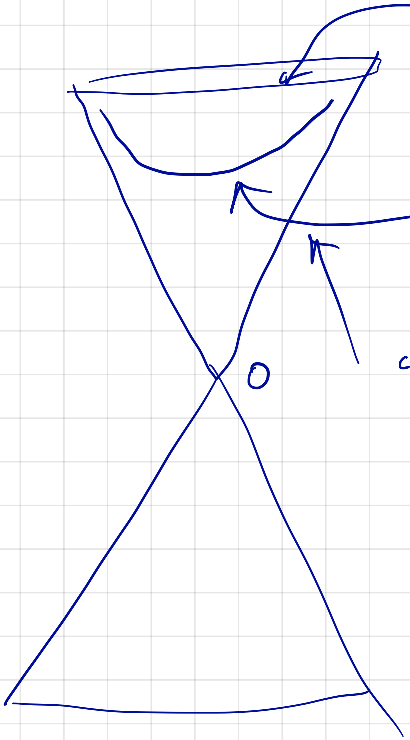
As in 2-d case, interval:

$$X = C(F - E)$$

$$\text{Int}(E,F) = X^T G X \qquad (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

physical Lorentz transformations preserve interval in physical coords

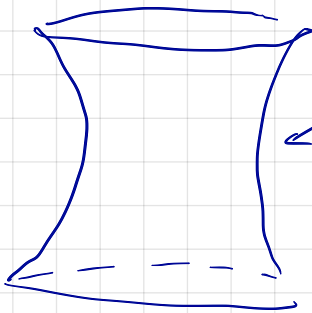
natural $\qquad$ natural coords




interval positive

interval $= K^2$ (proper time diff is $\frac{K}{c}$)

$c|\Delta t| = |\Delta \vec{x}|$, interval is $0$



hyperboloid for $\text{int}(O, P) = -1$

Call the things that are space distance 1 from origin

Def: The future light cone of origin is set of events $E = \begin{bmatrix} t \\ x \end{bmatrix}$ with $\mathrm{Int}(0, E) = 0$ and $t > 0$.

Def: A $4\times4$ matrix $L$ is a Lorentz transf, if

$$L^T G L = G.$$

a) It is orthochronous if in addition it sends the future light cone of the origin to itself.

b) It is proper if $\det(L) = 1$.

c) It is restricted if both proper + orthochronous

a) $O^+(1,3)$

b) $SO(1,3)$

c) $SO^+(1,3)$

Lemma: Let $L \in O(1,3)$. If $L$ is orthochronous then

$$L^0{}_0 > 0.$$

($L_{00}$; I'll explain why your book does this later)

Pf: $$L\begin{pmatrix}2\\0\\0\\0\end{pmatrix} = L\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + L\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix} ct_0 \\ * \end{bmatrix} + \begin{bmatrix} ct_1 \\ * \end{bmatrix}$$

with $t_0, t_1 > 0$. But $L\begin{pmatrix}2\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}2L^0{}_0 \\ * \end{pmatrix}$.

So $2L^0{}_0 = c(t_0 + t_1) > 0$.

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In fact, the converse is true; we'll wait a bit for this. But in the mean time, o.c. $\Leftrightarrow$ $L^0{}_0 > 0$ is fine.

Meaning of $L^0{}_0$:

$$L\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}L^0{}_0\\ L^1{}_0 \\ \vdots \\ L^3{}_0\end{bmatrix}$$

natural time coord of image of $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

e.g.
$$\mathcal{L}_x = \begin{bmatrix} c & s & 0 \\ s & c & \\ & 0 & I \end{bmatrix}$$

(boost in $x$–$t$ plane)

$c > 0$, det is $(c^2 - s^2) \cdot 1^2 = 1$.

e.g. $\begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \quad H \in SO(3)$

$$\begin{bmatrix} 1 & 0 \\ 0 & H^T \end{bmatrix} \underbrace{\begin{bmatrix} +1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & H^T \end{bmatrix} \begin{bmatrix} +1 & 0 \\ 0 & -H \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \checkmark$$

det is $1 \cdot \det(H) = 1$

$1 > 0$.

e.g.

$v$ any unit vector in $\mathbb{R}^3$

$K$ any element of $SO(3)$, $Ke_1 = v$

$$\left[ v \,\middle|\, \cdots \right]$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & K \end{bmatrix} \mathcal{L}_x \begin{bmatrix} 1 & 0 \\ 0 & K^T \end{bmatrix}$$

exercise: if $w \in \mathbb{R}^3$, $w \cdot v = 0$

$L \begin{bmatrix} 0 \\ w \end{bmatrix} = 0$. As for $v$, this is a boost in the $v-t$ plane.

More generally:

$$\begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \mathcal{L}_{?} \begin{bmatrix} 1 & 0 \\ 0 & K^T \end{bmatrix} \qquad H, K \in SO(3)$$

Exercise (next HW?)

$$L \in SO^+(1,3) \iff L = \begin{bmatrix} 1 & \\ & H \end{bmatrix} \Lambda_x \begin{bmatrix} 1 & \\ & K^T \end{bmatrix}$$

for some $\phi \in \mathbb{R}$, $H \in SO(3)$, $K \in SO(3)$.

They are compositions of spatial rotations and boosts.

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(4)- vectors