$$\frac{dx}{dt} = \frac{dx}{dt'} \cdot \frac{dt'}{dt} = \frac{\left[S + C\left(w/c\right)\right]c}{\left[C + S\left(w/c\right)\right]} = \frac{\left(S/C + \left(w/c\right)\right)c}{1 + \frac{S}{C}\frac{w}{c}}$$

$$= \frac{\left(\frac{v}{c} + \frac{w}{c}\right) \cdot c}{1 + \frac{v}{c}\frac{w}{c}}$$

$$= \frac{v + w}{1 + \frac{v}{c}\frac{w}{c}}$$

(Velocity addition formula again)

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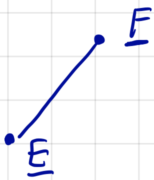



event

$F = (t_2, x_2)$

$E = (t_1, x_1)$

coordinates of the event in <u>same</u> inertial frame.

Interval: $c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 = I(E, F)$

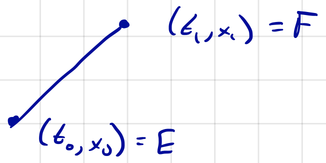
$$X = \begin{pmatrix} c(t_2 - t_1) \\ x_2 - x_1 \end{pmatrix} = \mathcal{C}\left(F - E\right)$$

$E = (t, x)$ physical coordinates

$(ct, x)$ natural coordinates (all entries units of length)

$$\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \quad \mathcal{C}$$

Given two points




I can construct a quantity associated with the pair called the <u>interval</u> separating them

1) $X = \mathcal{C}F - \mathcal{C}E = \mathcal{C}(F - E)$

(displacement from $E$ to $F$ in <u>natural</u> units

2) $G = \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$

can be $+, -, 0.$

$$\mathrm{Int}(E, F) = X^T G X$$

If $F - E = \begin{bmatrix} \Delta t \\ \Delta x \end{bmatrix}$, $\mathrm{Int}(E, F) = (c\Delta t)^2 - (\Delta x)^2$ units of length$^2$

Given another observer $O'$, what does $O'$ assign as the interval?

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = L_{-v} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$C \begin{bmatrix} t' \\ x' \end{bmatrix} = L_{-v} C \begin{bmatrix} t \\ x \end{bmatrix}$$

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \underbrace{C^{-1} L_{-v} C} \begin{bmatrix} t \\ x \end{bmatrix}$$

$L_{-v}$ : (natural) Lorentz transformation

physical Lorentz transformation.

$$CE' = L_{-v} CE$$

$$X' = C(E' - F') = L_{-v} C(E - F)$$

$$= L_{-v} X$$

$$(X')^T G X = X^T \underbrace{L_{-v}^T G L_{-v}} X$$

$$(X')^T G X = X^T \underbrace{L_{-v}^T G L_{-v}} X$$

$$\begin{pmatrix} c & -s \\ -s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c & -s \\ -s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix}\begin{pmatrix} c & -s \\ s & -c \end{pmatrix}$$

$$= \begin{pmatrix} c^2 - s^2 & 0 \\ 0 & -c^2 + s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= X^T G X.$$

The interval between two points, in any of these coordinate systems is

$$f(z) = C^{-1} L C z + T$$

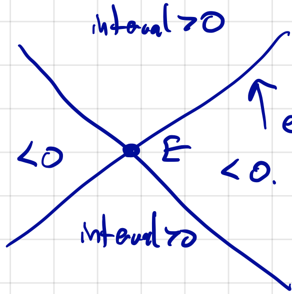
$$(c\Delta t)^2 - (\Delta x)^2$$

Of course, it is preserved by translations as well.

This is the fundamental quantity of 2-d spacetime that replaces the notion of distance in Euclidean geometry

Let us suppose $E$ is origin.

$$I(O,F) = 0 \quad \text{when } |\Delta x| = c|\Delta t|$$




exactly along paths of velocity c. "light cone"!

Where are all points with interval $= 1$ from $E = 0$?

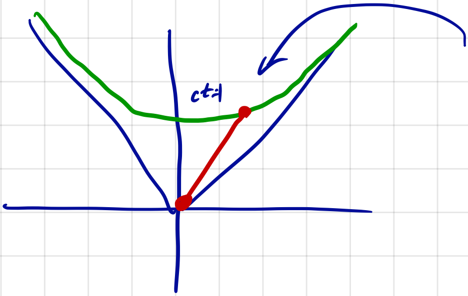
Here are some. Start with $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = X$. Now send through $\Lambda_\phi$.

$1 = ct$

$$\begin{bmatrix} c & s \\ s & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

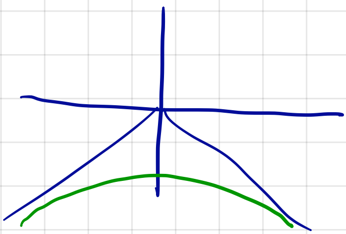
Recall: $c^2 - s^2 = 1$

$$(ct)^2 - x^2 = 1$$




this hyperbola.

We missed some: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ also has interval 1!

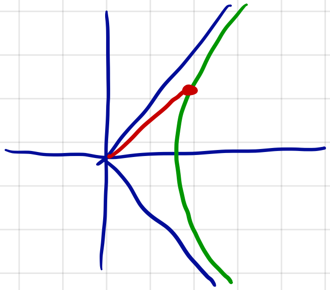


The transformations we are working with only take the upper branch to the upper branch, etc.

What about $-1$? Start with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

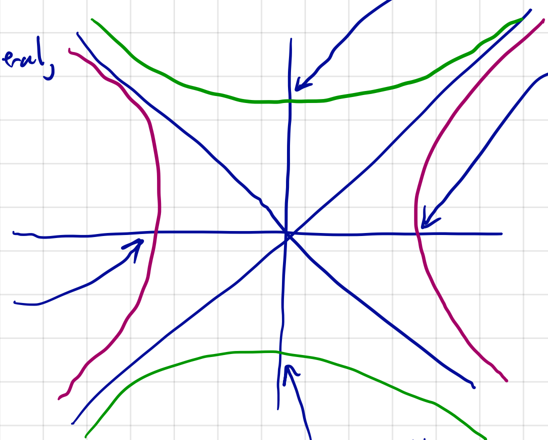
Now send through $L_\phi$ to get $\begin{bmatrix} s \\ c \end{bmatrix}$ $\quad s^2 - c^2 = -1$ ✓

$c > 0$ so always stays on right side.

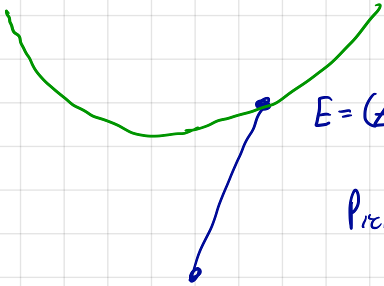


Similarly for left-hand branch.

In general,



So what does the interval measure?



$E=(t,x)$ $\quad ct^2 - x^2 = K^2 > 0, \quad t>0$

Pick $\psi$, $\tanh(\psi) = \dfrac{x}{ct}$

Exercise: $\quad \mathcal{L}_{-\psi}\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix}$

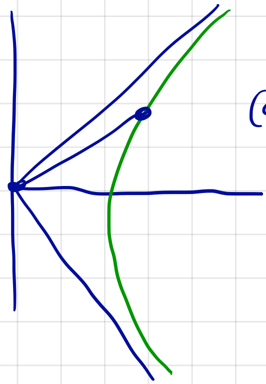

In a frame where $\underline{O}$ and $\underline{E}$ have the same space coordinate, $K/c$ is the time difference from $\underline{O}$ to $\underline{E}$.

In any other frame, the time difference is at least this big: it will be $\gamma \dfrac{K}{c}$ [time dilation, the companion of length contraction]

Similar if $t<0$.

$K/c$

We call this the "proper time" separating $O$ and $E$



$$(ct)^2 - x^2 = -K^2 < 0 \qquad x > 0$$

$$\tanh \phi = \frac{ct}{x}$$

exercise: $\mathcal{L}_{-\phi}\begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ K \end{bmatrix}$

$K$ is the distance between $\underline{O}$ and $\underline{E}$ in a coordinate system in which $\underline{O}$ and $\underline{E}$ are simultaneous.

We call this the proper distance between $\underline{O}$ and $\underline{E}$.

The full set of transformations of $\mathbb{R}^2$ that preserve the interval is known as the Poincaré group.

Those that preserve the origin are called the Lorentz group. $O(1,1)$

There are some weirdos:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{(time reflection)}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} = \begin{bmatrix} -ct \\ x \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{space reflection.}$$

We can rule these out via $\det(A) = 1$.

$SO(1,1)$.

But this still allows $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (space + time reflection)

$SO^+(1,1)$ for taking future to future

Now 3 space dimensions.

We'll assume coordinate transformations are affine (so non-acceleration is preserved).

We'll assume that if one observer says $\underline{E}$ and $\underline{F}$ lie on the path of a non accelerating particle traveling at the speed of light, all observers say so.

$$\begin{matrix} E = (t_0, x_0) \\ F = (t_1, x_1) \end{matrix} \rightarrow \quad |c\Delta t| = |\Delta x|$$

Let us deal with linear case first

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = A \begin{pmatrix} t \\ x \end{pmatrix}$$

$$\downarrow$$

$$C^{-1} L C$$

$$C = \begin{bmatrix} c & \\ & 1 \end{bmatrix}$$

$F$ lies on path of a photon from $O$:

$$c|\Delta t| = |x|$$

$$c^2(\Delta t)^2 = |x|^2$$

$$(CF)^T G\, CF = 0 \qquad G = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$$

28: phys
30 age age
redo: sem

We require

$$(CAF)^T G\, CAF = 0$$

whenever $(CF)^T G\, CF = 0$

$$A = C^{-1} L C \qquad (CF)^T L^T G\, L CF = 0$$

whenever

I.e. $\quad X^T L^T G L X = 0$ whenever

$$X^T G X = 0$$

Full spacetime:

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ spacetime metric.

Interval between $(t_1, x_1, y_1, z_1) = E_1$ is $(t_2, x_2, y_2, z_2) = E_2$

$$[c(t_2 - t_1)]^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

$$(E_2 - E_1)^T C^T G \underbrace{C (E_2 - E_1)}_{X}$$

$$X^T G X$$

A Lorentz transformation is a linear map $L$