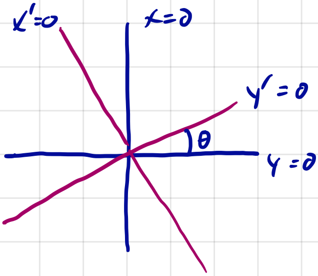
Last class:

2-d Lorentz transforms

O' moving with velocity v relative to O, intersect at

$t=0 \quad x=0 \qquad (t'=0,\ x'=0)$.

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$



passive (relabeling)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(e.g. $y'<0$ if $x>0 \quad y=0$)

(active)

On the other hand, the rotation that takes the xy axes to the x', y' axis is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

I want a similar picture for Lorentz transformations.

$$L_v = \gamma \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

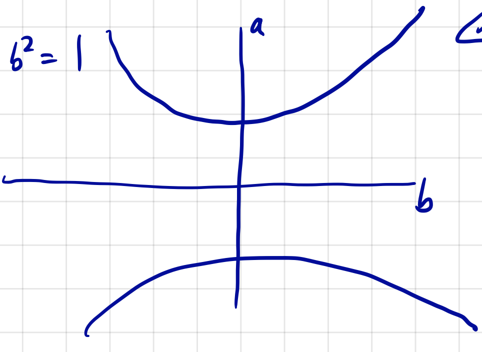
(transformation that plunks O's configuration onto O''s)

$$a^2 - b^2 = \gamma^2 \left[1 - \left(\frac{v}{c}\right)^2 \right]$$

$$= \frac{1}{1-\left(\frac{v}{c}\right)^2} \left(1 - \left(\frac{v}{c}\right)^2\right) = 1$$

[these maps are "area" preserving. Spacetime has a notion of area or volume]

$a^2 - b^2 = 1$



hyperbola.

But $a > 0$ so just

Moreover, you are familiar with $\cos^2\theta + \sin^2\theta = 1$.

If $a^2+b^2=1$ there is a unique $\theta \in [0, 2\pi)$ with $a = \cos\theta$, $b = \sin\theta$.

Recall the hyperbolic trig functions:

$$\cosh z = \frac{e^{z}+e^{-z}}{2} \qquad \sinh z = \frac{e^{z}-e^{-z}}{2}$$

Exercise: $\cosh^2 z - \sinh^2 z = 1$

In fact, if $a^2-b^2=1$ there is a unique $\psi \in \mathbb{R}$

$$a = \cosh\psi$$

$$b = \sinh\psi$$

In fact: $\frac{b}{a} = \tanh(\psi)$, $\psi = \operatorname{arctanh}(b/a)$

So $\gamma = \cosh(\psi)$ $\quad$ ψ rapidity.

Recall the angle sum formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i(\theta_1+\theta_2)} = \cos\theta_1+\theta_2 + i\sin(\theta_1+\theta_2)$$

$$\downarrow$$

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= [\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2]$$

$$+ i[\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1]$$

Exercise:

$$\cosh(z_1+z_2) = \cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$$

$$\sinh(z_1+z_2) = \sinh(z_1)\cosh(z_2) + \sinh(z_2)\cosh(z_1)$$

(Do this from the definition).

$$\mathcal{L}_\phi = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$$

$$\mathcal{L}_{\phi_2}\,\mathcal{L}_{\phi_1} = \begin{pmatrix} C_2 & S_2 \\ S_2 & C_2 \end{pmatrix}\begin{pmatrix} C_1 & S_1 \\ S_1 & C_1 \end{pmatrix}$$

$$= \begin{pmatrix} C_1C_2 + S_1S_2 & C_2S_1 + S_2C_1 \\ C_2S_1 + S_2C_1 & C_1C_2 + S_1S_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\phi_1+\phi_2) & \sinh(\phi_1+\phi_2) \\ \sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) \end{pmatrix}$$

$$= \mathcal{L}_{\phi_1+\phi_2}$$

The Lorentz matrices are closed under mult.

They contain the id, $\mathcal{L}_0$. $\mathcal{L}_{-\phi} = \mathcal{L}_\phi^{-1}$ by above.

So they form a group.

$SO^+(1,1)$ ($\underbrace{\text{proper, orthochronous}}_{\text{we'll see why shortly.}}$ Lorentz transformations)

Suppose O'' is traveling with velocity w relative to O', and O' is traveling with velocity v relative to O and all agree on a common origin.

What is the transformation from O to O'' coordinates?

$$\psi_1 = \operatorname{arctanh}\left(-\frac{v}{c}\right)$$

$$\psi_2 = \operatorname{arctanh}\left(-\frac{w}{c}\right)$$

$$\begin{pmatrix} t'' \\ x'' \end{pmatrix} = \begin{pmatrix} C_2 & S_2 \\ S_2 & C_2 \end{pmatrix}\begin{pmatrix} C_1 & S_1 \\ S_1 & C_1 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}$$

$$= \begin{pmatrix} C_3 & S_3 \\ S_3 & C_3 \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} \qquad \begin{aligned} C_3 &= \cosh(\psi_1+\psi_2) \\ S_3 &= \sinh(\psi_1+\psi_2) \end{aligned}$$

So O'' is moving w.r.t. O with velocity z

$$\tanh(\psi_1 + \psi_2) = \left(\frac{-z}{c}\right)$$

$$-\frac{z}{c} = \tanh(\psi_1 + \psi_2) = \frac{\sinh(\psi_1 + \psi_2)}{\cosh(\psi_1 + \psi_2)}$$

$$= \frac{S_1 C_2 + S_2 C_1}{C_1 C_2 + S_1 S_2}$$

$$= \frac{S_1/C_1 + S_2/C_2}{1 + \frac{S_1}{C_1}\frac{S_2}{C_2}}$$

$$\frac{S_1}{C_1} = \tanh(\psi_1) = \frac{-v}{c}$$

$$\frac{S_2}{C_2} = \tanh(\psi_2) = -\frac{w}{c}$$

$$\frac{z}{c} = \frac{\frac{v}{c} + \frac{w}{c}}{1 + \left(\frac{v}{c}\right)\left(\frac{w}{c}\right)}$$

addition of velocity

but tempered

exercise: if $a, b \in (-1, 1)$

$$\frac{a+b}{1+ab} \in (-1,1) \text{ also.}$$

$a+b \leq 1+ab$

$1 + \frac{b}{a} \leq \frac{1}{a} + b \quad (a>0)$

$\frac{b-1}{a} \leq b-1$ ✓

Similarly, suppose O' is moving with velocity v w.r.t. O

And it observes a particle traveling with velocity w.

What velocity does O observe?

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} ct' \\ wt' + x_0' \end{pmatrix} \qquad \text{assuming at } x_0' \text{ when } t'=0$$

$$= \begin{pmatrix} ct' \\ \left(\frac{w}{c}\right) ct' + x_0' \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} ct' \\ \left(\frac{w}{c}\right) ct' + x_0' \end{pmatrix}$$

What matters: $\begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} ct' \\ \frac{w}{c} ct' \end{pmatrix} = \begin{matrix} \left[C + S\left(\frac{w}{c}\right)\right] ct' \\ \left[S + C\left(\frac{w}{c}\right)\right] ct' \end{matrix}$

$$\downarrow$$

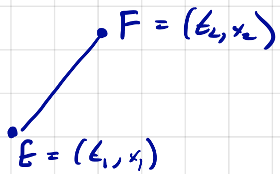
$$\begin{pmatrix} C & S \\ S & C \end{pmatrix} L_w \begin{pmatrix} 1 \\ 0 \end{pmatrix} ct' = L_v L_w \begin{pmatrix} 1 \\ 0 \end{pmatrix} ct'$$

$$\frac{dx}{dt} = \frac{dx}{dt'} \cdot \frac{dt'}{dt} = \frac{[S + C(w/c)]\,c}{[C + S(w/c)]} = \frac{\left(S/C + (w/c)\right) c}{1 + \frac{S}{C}\frac{w}{c}}$$

$$= \frac{\frac{v}{c} + \frac{w}{c}}{1 + \frac{v}{c}\frac{w}{c}} \cdot c$$

$$= \frac{v + w}{1 + \frac{v}{c}\frac{w}{c}}$$

(Velocity addition formula again)



Interval: $c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 = I(E, F)$

$$X = \begin{pmatrix} c(t_2 - t_1) \\ x_2 - x_1 \end{pmatrix} = C\,(F - E)$$

Interval: $\quad X^T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{G} X$

Coordinates for O' $\quad \begin{pmatrix} ct' \\ x' \end{pmatrix} = L_{-v} \begin{pmatrix} ct \\ x \end{pmatrix}$

$$X' = L_{-v} X$$

$$(X')^T G X = X^T \underbrace{L_{-v}^T G L_{-v}}\, X$$

$$\begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & c \end{pmatrix} \begin{pmatrix} c & -s \\ s & -c \end{pmatrix}$$

$$= \begin{pmatrix} c^2 - s^2 & 0 \\ 0 & -c^2 + s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= X^T G X.$$

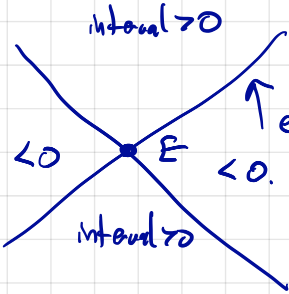
The interval between two points, in any of these coordinate systems is

$$(c\Delta t)^2 - (\Delta x)^2$$

This is the fundamental quantity of 2-d spacetime that replaces the notion of distance in Euclidean geometry.

Let us suppose E is origin.

$$I(O,F) = 0 \quad \text{when} \quad |\Delta x| = c|\Delta t|$$



exactly along paths of velocity c. "light cone"

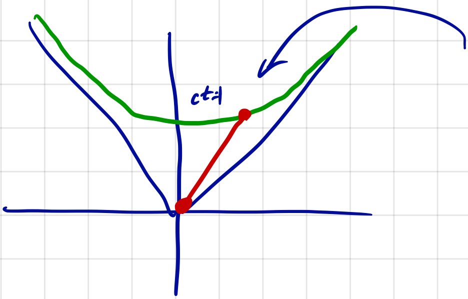
Where are all points with interval $= 1$?

Here are some. Start with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ($1 = ct$). Now send through Λ_ϕ.

$$\begin{bmatrix} c & s \\ s & c \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ s \end{bmatrix}$$

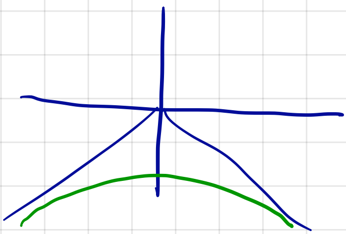
Recall: $c^2 - s^2 = 1$

$$(ct)^2 - x^2 = 1$$



this hyperbola.

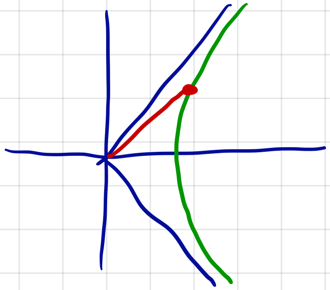
We missed some: $\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ also has interval 1!



The transformations we are working with only take the upper branch to the upper branch, etc.

What about -1? Start with $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now send through $\mathcal{L}_\phi$ to get $\begin{bmatrix} s \\ c \end{bmatrix}$ $\quad s^2 - c^2 = -1$ ✓



$c > 0$ so always stays on right side.

Similarly for left-hand branch.