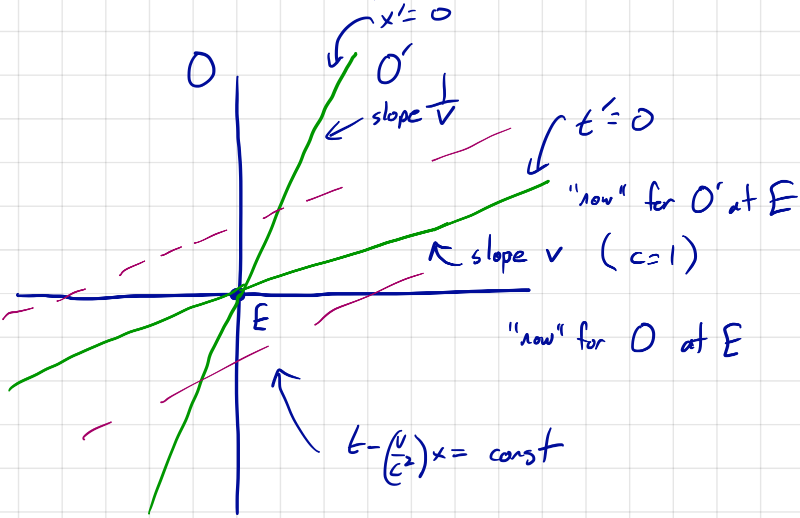
i.e. $t = \left(\frac{v}{c^2}\right) x$

It is handy to draw pictures with units $c = 1$.

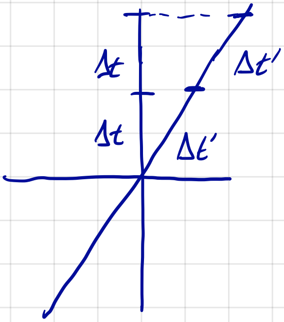


Repeat at different spots on O''s world line and get

$$t - \left(\frac{v}{c^2}\right) x = \text{const} \iff t' = \text{const}$$

$$t' = f\left(t - \left(\frac{v}{c^2}\right) x\right)$$

What's f?



$$\Delta t' = \alpha \Delta t \quad \text{when } x' = 0 \quad (\text{i.e. } x = vt)$$

$$\Downarrow$$

$$t' = \alpha t \quad \text{when } x' = 0$$

$$t' = f\left(t - \frac{v}{c^2}x\right)$$

$$\Downarrow$$

$$\alpha t = f\Big(\underbrace{t\left(1 - \left(\tfrac{v}{c}\right)^2\right)}_{s}\Big) \qquad [x = vt]$$

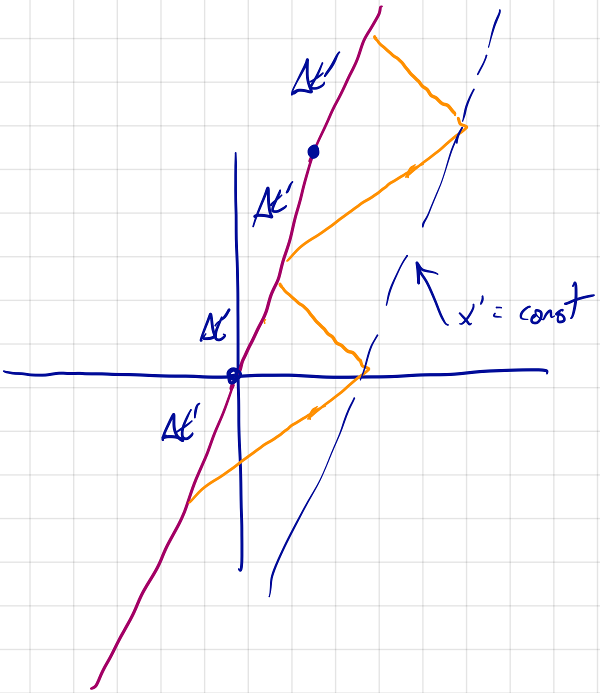
$$\frac{\alpha s}{1 - \left(\frac{v}{c}\right)^2} = f(s)$$

I.e. we've identified f up to α.

$$f(s) = \gamma s \quad \text{for some } \gamma = \frac{\alpha}{1 - \left(\frac{v}{c}\right)^2}$$

$$t' = \gamma\left[t - \frac{v}{c^2}x\right]$$

Similarly, x' is constant on lines parallel to $x = vt$



$$x' = g(x - vt)$$

Value of x' is determined from value when $t' = 0$.

$$\Downarrow$$

$$t = \frac{v}{c^2} x$$

$$x' = g\left(x\left(1 - \left(\tfrac{v}{c}\right)^2\right)\right)$$

on the line $t' = 0$.

Claim: $x' = \beta x$ on line $t' = 0$ for some β.

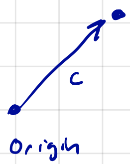
Assuming this, since $t' = 0 \iff t - \frac{v}{c^2} x = 0$

$$\beta x = x' = g(x - vt) = g\left(x\left(1 - \left(\tfrac{v}{c}\right)^2\right)\right)$$

$$\Rightarrow \quad g(s) = \underbrace{\frac{\beta}{1 - \left(\frac{v}{c}\right)^2}}_{\gamma \text{ for now.}} s$$

I.e. $\quad x' = g(x-vt) = \hat{\gamma}(x-vt)$

Moreover: $\quad \hat{\gamma} = \gamma$



(t, ct) O - coord

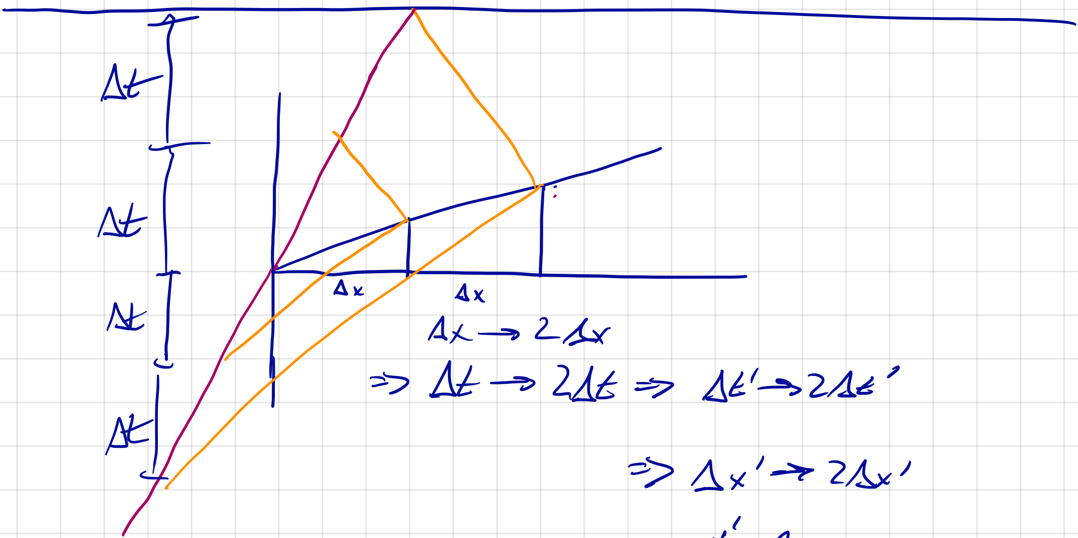
(t', ct') O' - coords

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) = \gamma t\left(1 - \frac{v}{c}\right)$$

$$x' = \hat{\gamma}(x - vt) = \hat{\gamma}(ct - vt)$$

$$= \hat{\gamma}\, ct\left(1 - \frac{v}{c}\right)$$

$$x' = ct' \Rightarrow \hat{\gamma} = \gamma.$$



$\Delta x \to 2\Delta x$

$\Rightarrow \Delta t \to 2\Delta t \Rightarrow \Delta t' \to 2\Delta t'$

$\Rightarrow \Delta x' \to 2\Delta x'$

$\Rightarrow x' = \beta x$

What's the value of γ?

$$x' = \gamma (x - vt) \qquad\qquad x = \gamma' (x' + vt')$$

$$t' = \gamma \left(t - \frac{v}{c^2} x\right) \qquad\qquad t = \gamma' (x' + vt')$$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \gamma \begin{bmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \qquad \begin{pmatrix} t \\ x \end{pmatrix} = \gamma' \begin{bmatrix} 1 & \frac{v}{c^2} \\ v & 1 \end{bmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$= \gamma \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma' \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$I = \gamma\gamma' \cdot \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \left(\frac{v}{c}\right)^2 & 0 \\ 0 & 1 - \left(\frac{v}{c}\right) \end{pmatrix}$$

Reasonable symmetry: γ depends on $|v|$ only, so $\gamma = \gamma'$!

$$\gamma^2 = \frac{1}{1 - \left(\frac{v}{c}\right)^2} \Rightarrow \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

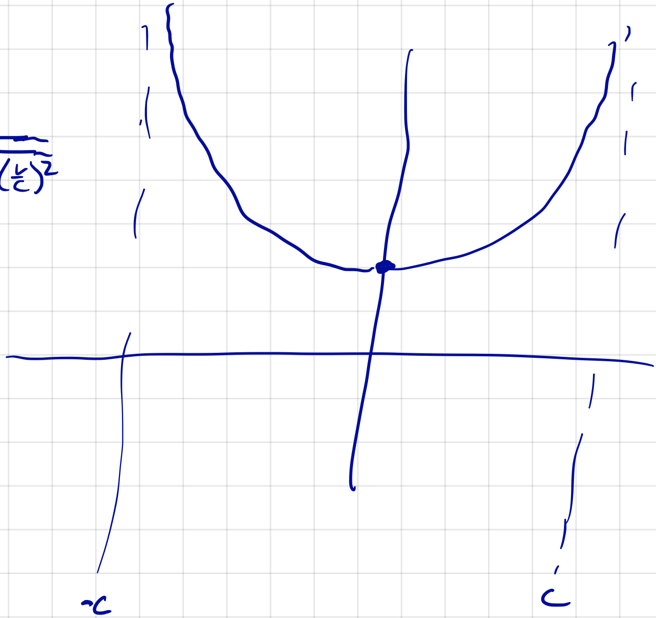
$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

So:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & v/c \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v/c \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

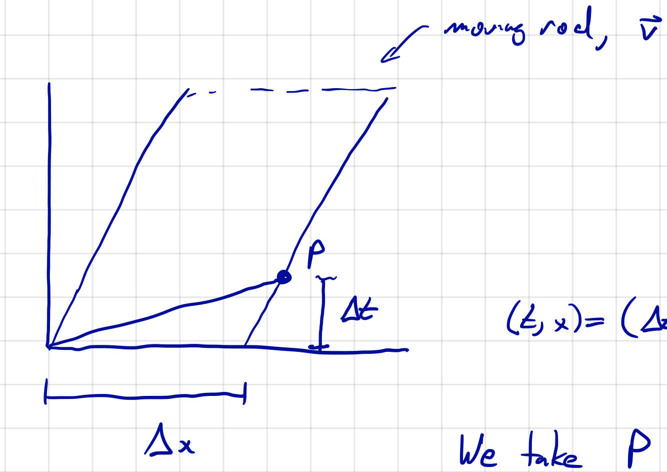
$$\gamma : \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$



$\gamma \geq 1 \quad \gamma \to \infty$ as $v \to \pm c$

$\gamma = 0 \iff v = 0.$

Two consequences.



$$(t, x) = (\Delta t, \Delta x + v\Delta t)$$

We take P to lie on $t' = 0$ so as to measure length w.r.t. O'

$$t' = 0 \iff t - \frac{v}{c^2} x = 0$$

$$P = \left(\frac{v}{c^2} x, x\right) \text{ for some } x$$

$$P = (\Delta t, \Delta x + v\Delta t) \text{ for some } \Delta t$$

$$\Delta t = \frac{v}{c^2} x$$

$$x = \Delta x + \frac{v^2}{c^2} x$$

$$\gamma^{-2} x = \Delta x$$

$$x = \gamma^2 \Delta x$$

note sign

$$x' = \gamma(x - vt)$$

$$= \gamma\left(x - \frac{v^2}{c^2}x\right)$$

$$= \gamma\,\gamma^{-2}x$$

$$= \gamma\,\Delta x$$

I.e. $\Delta x' = \gamma\,\Delta x$

So the rod in the rest frame is longer, by a factor of γ, than the rod in the moving frame.

The rod in the moving frame is shorter by a factor of $1/\gamma$.

This is known as length contraction.