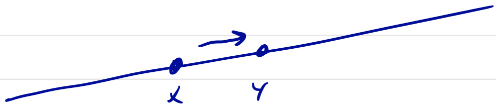
Orientation:

1-d: $x, y \in \mathbb{R}$

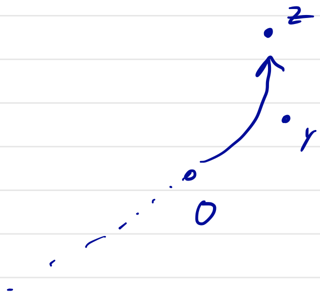
(x, y) is positively oriented if $y > 0$.



$(0, z)$ is positively oriented if $z > 0$

(x, y) is positively oriented $\Leftrightarrow$ $(0, y-x)$ is.

2-d. $(0, y, z) \in \mathbb{R}^2$ positively oriented $\Leftrightarrow$



$$\det \begin{pmatrix} y_1 & z_1 \\ y_2 & z_2 \end{pmatrix} > 0 \qquad \vec{y} \times \vec{z} > 0$$

intuitively, if z is on left side of $(0, y)$

(y, y, z) is positively oriented $\Leftrightarrow$

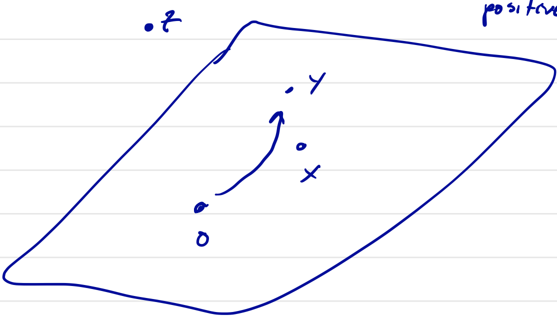
$(0, y-x, z-x)$ is.

3-d: $(0, x, y, z)$ is positively oriented if

$$\det(x, y, z) > 0$$

(w, x, y, z) is positively oriented if $(0, x-w, y-w, z-w)$ is.

positively oriented if on this side



Exercise: Suppose $A \in O(3)$.
Then $A \in SO(3)$ iff whenever

(x, y, z) is positively oriented, so is (Ax, Ay, Az).

cols

Matrix mult: $B = (x \;\; y \;\; z)$

cols

$$AB = (Ax \;\; Ay \;\; Az)$$

Galilean Relativity

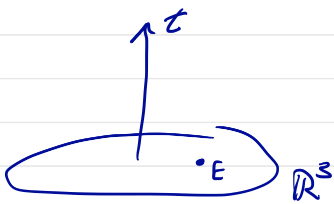
$\mathcal{S}$: spacetime, the thing, it abides

$$\begin{tikzcd} \mathcal{S} \arrow[d, "\phi"'] \\ \mathbb{R}\times\mathbb{R}^3 \end{tikzcd}$$

just like distance can be easily computed in preferred coords, there are properties of $\mathcal{S}$ that can be detected in preferred coordinates.

$E = (t, x) \in \mathbb{R}\times\mathbb{R}^3$, an event

$E_1 = (t_1, x_1)$, $E_2 = (t_2, x_2)$



a) simulenaity

two events are <u>simulaneous</u> if $t_1 = t_2$

b) time orientation: E_2 is <u>later</u> than E_1 if $t_2 > t_1$

c) time scale: absolute time difference is $|t_2 - t_1|$

d) dist between simultaneous events: If $t_1 = t_2$, distance is

$\langle x_2 - x_1, x_2 - x_1 \rangle$.

e) non-acceleration

A free particle is a curve

$\gamma : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3$ of the form

$$\gamma(s) = (s + t_0, x_0 + vs)$$

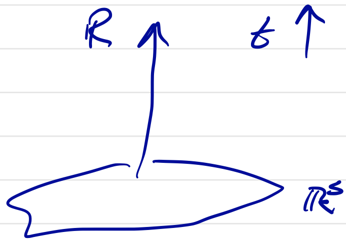
$$t_0 \in \mathbb{R} \quad x_0 \in \mathbb{R}^3 \quad v \in \mathbb{R}^3$$

$t(s) = (s + t_0)$ (just a translation)

$x(s) = x_0 + vs$ $(\Leftrightarrow x''(s) = 0)$

Galilean Relativity

Arena of Physics $\underbrace{\mathbb{R}}_{\text{time}} \times \underbrace{\mathbb{R}^3}_{\text{space.}}$



$(t, x) \leftarrow$ events

1) Two events are <u>simultaneous</u> if $t_1 = t_2$

$E_1 = (t_1, x_1)$ $E_2 = (t_2, x_2)$

2) The event E_2 is <u>later</u> than E_1 if $t_2 > t_1$

3) The <u>absolute time difference</u> between E_1 and E_2 is $|t_2 - t_1|$

4) If E_1 and E_2 are simultaneous, the <u>distance</u> between E_1 and E_2 is $\|x_2 - x_1\|$ → Euclidean norm.

[Note: we do not assign dist between non-simul events]

5) A curve $\gamma : \mathbb{R} \to \mathbb{R} \times \mathbb{R}^3$ is <u>non-accelerating</u> if it has the form $\gamma(t) = (t + t_0,\ x_0 + tv)$ $x_0, v \in \mathbb{R}^3,\ t_0 \in \mathbb{R}$.

An map $F: \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}\times\mathbb{R}^3$ is called a Galilean transformation if it preserves properties 1) – 5)

i.e., takes simultaneous events to simultaneous events, preserves distance between simultaneous events, etc.

What are the Galilean transformations?

$$f(t,x) = (T(t,x), X(t,x)).$$

Note: $f(t,x_1)$ is simultaneous with $f(t,x_2)$ $\forall t, x_1, x_2$

so $T(t,x_1) = T(t,x_2)$ $\forall t, x_1, x_2 \Rightarrow$

T is a function of t alone.

Let $t_0 = T(0)$.

If $t \geq 0$ a) $T(t) > T(0)$ (by time orientation) obvious

b) $\underbrace{T(t) - T(0) = |T(t) - T(0)|}_{a)} \underbrace{=}_{\text{preservation of time interval}} |t - 0| = t$

$\Rightarrow T(t) = t + t_0$

Ditto for $t < 0$.

For each fixed t

$X(t,x)$ is an isometry of $\mathbb{R}^3$.

So $\quad X(t,x) = H(t)\,x + T(t) \qquad H \in O(3)$
$T \in \mathbb{R}^3$

for each t.

Now let $\gamma(s) = (t_1, x_1) + (s, ws)$

be a non accelerating curve.

So is $f \circ \gamma$?

$$f \circ \gamma = \Big(\underbrace{t_1 + t_0 + s}_{\text{right form}},\ \underbrace{H(t_0+s)(x_1 + ws) + T(t_0+s)}_{\text{needs to have vanishing 2nd derivative}} \Big)$$

$$H''(x_1 + vs) + 2H'v + 0 + T'' = 0.$$

$x_1,\ v = 0 \Rightarrow T'' = 0 \Rightarrow T = x_0 + vt$

$v = 0,\ x_1$ arbitrary $\Rightarrow H'' = 0$

Case 1: take $x_1 = 0$, $w = 0$, $t_1 = 0$

$$T''(s) = 0 \implies T(s) = x_0 + vs$$

for some $x_0 \in \mathbb{R}^3$,
some $v \in \mathbb{R}^3$

Case 2: $w = 0$, $t_0 = 0$

$$\left(\frac{d}{ds}\right)^2 H(s)\, x_1 = 0$$

$x_1 = e_1, \dots, e_n \implies$

each col of $H(s)$ is twice diff, as is H, and $H'' = 0$

Case 3: $t_0 = 0$, $x_1 = 0$

$$0 = \left(\frac{d}{ds}\right)^2 \left[H(s)\, w\, s\right] = H''(x_1 + ws) + 2H'w$$

$$= 2H'w$$

Since w is arbitrary H is constant.

So $H'v = 0 \quad \forall v.$

$\Rightarrow H' = 0$

$\Rightarrow H$ is constant.

Galelam Transformation:

$$f(t,x) = (t + t_0,\ Hx + y_0 + vt)$$

$$= (t,\ Hx + vt) + (t_0, y_0)$$

$$= \begin{bmatrix} 1 & 0 \ 0 \ 0 \\ v & H \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} + \begin{bmatrix} t_0 \\ y_0 \end{bmatrix}$$

$v \in \mathbb{R}^3$,
$t_0 \in \mathbb{R}$
$v_0 \in \mathbb{R}^3$
$H \in O(3)$

If, in addition, $H \in SO(3)$

we call this a <u>proper</u> Galelam transformation.

Exercise: the Galilean transformations form a group.

Picture: Spacetime in classical mechanics admits preferred coordinate systems ("Galilean coordinates")

Transformations between these coordinate systems are Galilean transformations, which are precisely the maps that preserve

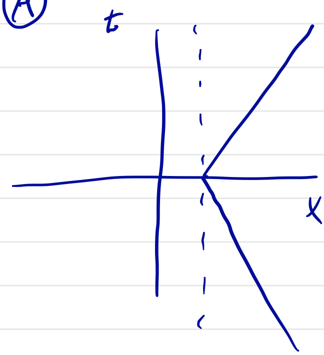
a) simultaneity
b) time orientation
c) spatial distance between simultaneous events
d) non-accelerating curves

Physicists call these coordinate systems inertial frames.

Important: The <u>family</u> of coordinate systems is preferred, but no particular system is.

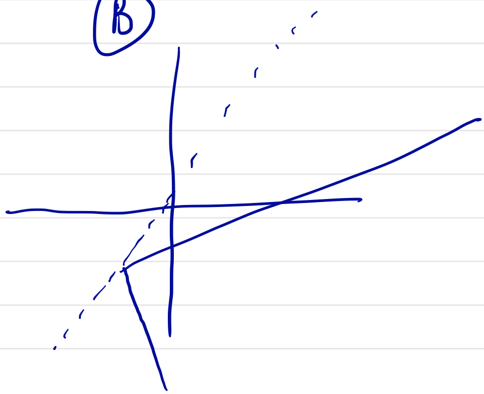
"Galilean relativity"

Ⓐ



vs

Ⓑ



(A) and (B) represent the same physical process.

There is a Galilean transformation taking (A) to (B).

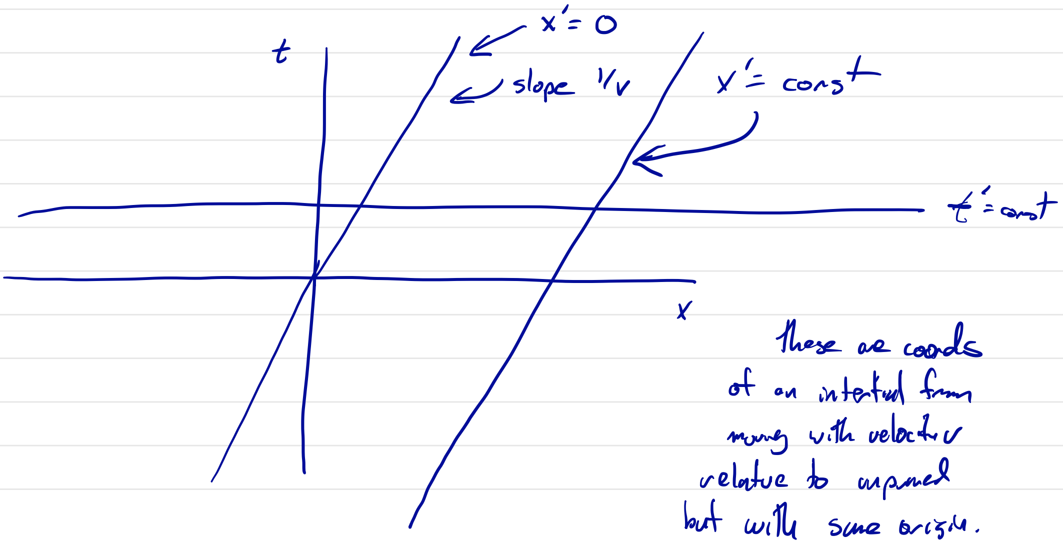
Special category: boosts

$A = I$

$y_0 = 0, \quad t_0 = 0 \qquad \vec{v} = (v, 0, 0)$

$t = t'$

$x = x' + vt' = x' + vt \qquad\qquad t = \frac{1}{v} x$



these are coords of an inertial frame moving with velocity v relative to unprimed but with same origin.

Other special categories

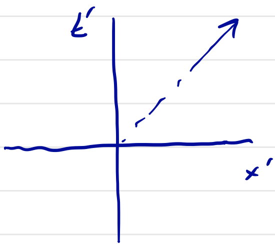
Translations: $(t,x) \longmapsto (t+t_0, x+x_0)$

Spatial rotations: $(t,x) \longmapsto (t, Hx)$.

Every Galilean transformation is a composition of these three, in the same sense that every isom is a comp of a rotation and a translation.

Alas, the Galilean picture is not the universe we live in.

Observed fact: the speed of light is the same for all observers. Contradicts Galilean relativity



c = speed of light, particle traveling with speed c in pos. x' dir.

$$x'(t') = ct' \qquad t' = t$$

$$x(t) = x'(t) + vt$$
$$= ct + vt$$
$$= (c+v)t$$

$\downarrow$

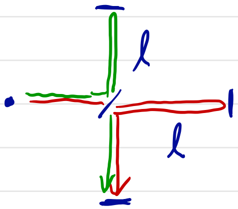
not c!

How? Michelson-Morley '89

Premise: Electromagnetic waves are waves in something: aether

Earth is moving with respect to aether.

$\xrightarrow{v}$ (w.r.t. aether)



$c-v \longrightarrow$

$c+v \longleftarrow$

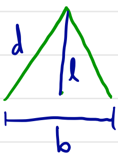
time for red out and back

$$\frac{\ell}{c+v} + \frac{\ell}{c-v} = \frac{2c\ell}{c^2-v^2}$$

$$= \frac{2\ell}{c}\left[\frac{1}{1-\left(\frac{v}{c}\right)^2}\right]$$

$$= \frac{2\ell}{c}\left[1 + \left(\frac{v}{c}\right)^2 + O\left(\frac{v}{c}\right)^4\right]$$

Green: in restframe of aether



Time for one leg: $\frac{d}{c}$

Time for both legs $\frac{2d}{c}$

Distance of b: $\frac{2d}{c} \cdot v = 2d\left(\frac{v}{c}\right)$

$$d^2 = \ell^2 + d^2\left(\frac{v^2}{c^2}\right)$$

$$\frac{d}{dx}\frac{1}{\sqrt{1-x}} = -\frac{1}{2}(1-x)^{-3/2}\cdot(-1)$$

$$= \frac{1}{2}(1-x)^{-3/2}$$

$$d^2\left[1-\left(\frac{v}{c}\right)^2\right] = \ell^2$$

$$d\sqrt{1-\left(\frac{v}{c}\right)^2} = \ell$$

$$\frac{2d}{c} = \frac{2\ell}{c}\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

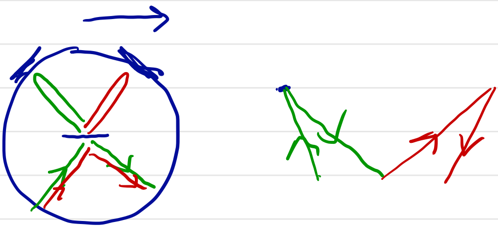
$$= \frac{2\ell}{c}\left[1+\frac{1}{2}\left(\frac{v}{c}\right)^2 + O\left(\left(\frac{v}{c}\right)^4\right)\right]$$

$$= \frac{2\ell}{c} + \frac{\ell}{c}\left(\frac{v}{c}\right)^2 + \frac{2\ell}{c}O\left(\left(\frac{v}{c}\right)^4\right)$$

Time difference: red - green $= \frac{\ell}{c}\left(\frac{v}{c}\right)^2 + \frac{2\ell}{c}O\left(\left(\frac{v}{c}\right)^4\right)$

↑ small compared with

But



By symmetry, path lengths, and time are same.

Waves add constructively:

$$\sin(\omega t) + \sin(\omega t) = 2\sin(\omega t)$$

$$\sin(\omega t + \pi) + \sin(\omega t) = 0$$

leads to interference patterns

So the difference in travel time should cause different inteference patterns for the two configurations, and should smoothly transition as apparatus is rotated.

Instead: pattern is independent of orientation of apparatus.

Conclusion: $v = 0$ w.r.t. aether.

(Full Aether drag)

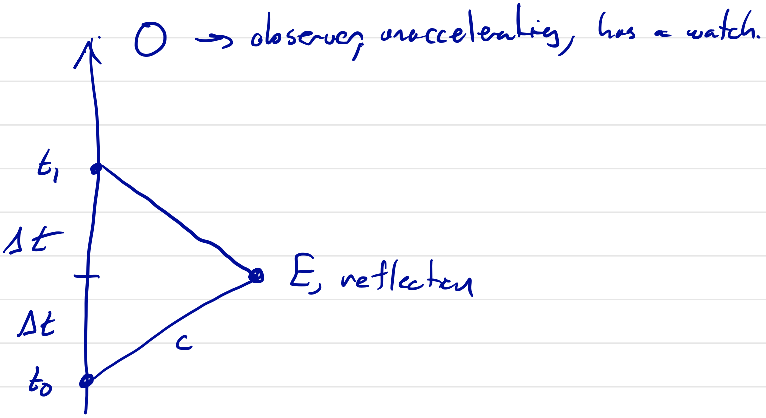
Inconsistent with two other experiments

stellar aberration: suggests no aether drag

Fizeau: consistent with partial aether drag

Way out: Light travels with velocity c for all inertial observers. (So in rest frame of experiment, no time difference as apparatus is rotated).

How to put coordinates on spacetime (radar method)

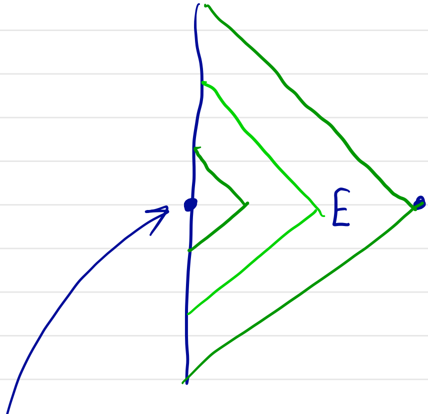


By symmetry E has time coordinate $\dfrac{t_0 + t_1}{2}$.

How far away? $t_1 - t_0 = 2\Delta t$

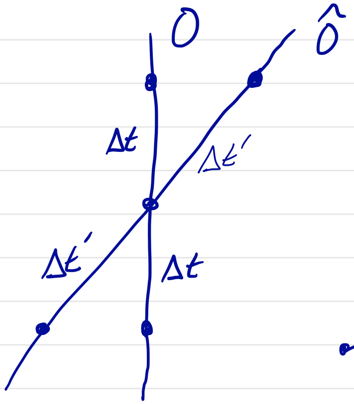
$$\Delta x = c\,\Delta t = \frac{t_1 - t_0}{2}\,c$$

What are events simultaneous with E?



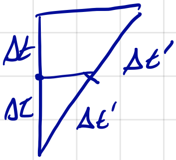
midway time between send/receive is the same.

Now consider O' traveling with velocity v relative to O

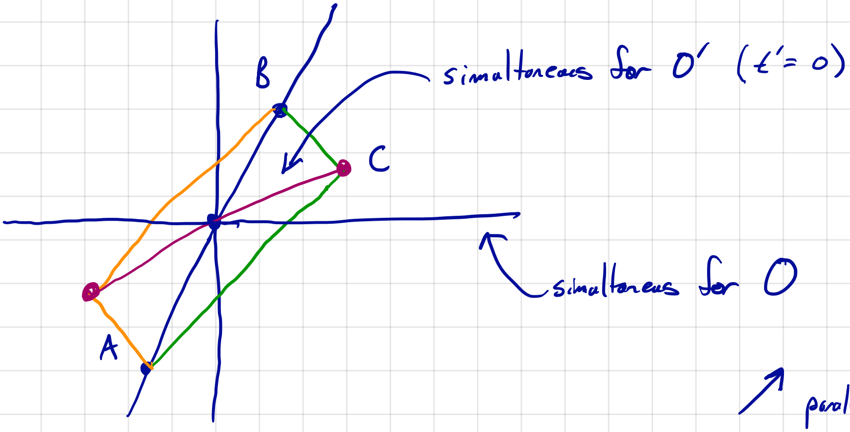


(Translation invariance of time differences)

I



Now suppose O' does radar:



$\nearrow$ parallel to $(1, c)$

For O: $A = (-\Delta t, -v\Delta t)$
$B = (\Delta t, v\Delta t)$

$$C = A + \lambda_A (1, c)$$
$$= B + \lambda_B (1, -c)$$

OMIT

$$-\Delta t + \lambda_A = \Delta t + \lambda_B \Rightarrow \lambda_A - \lambda_B = 2\Delta t$$

$$-v\Delta t + \lambda_A c = v\Delta t - \lambda_B c = \quad (\lambda_A + \lambda_B)c = 2v\Delta t$$

$$\lambda_A + \lambda_B = 2\left(\frac{v}{c}\right)\Delta t$$

Add: $2\lambda_A = 2\Delta t\left[1 + \left(\frac{v}{c}\right)\right]$

$$\lambda_A = \Delta t\left[1 + \left(\frac{v}{c}\right)\right]$$

$$\lambda_B = \lambda_A - 2\Delta t = \Delta t\left[-1 + \left(\frac{v}{c}\right)\right]$$

$$C = (-\Delta t, -v\Delta t) + \Delta t\left[1 + \left(\frac{v}{c}\right)\right](1, c)$$

$$= \left(\Delta t\left(\frac{v}{c}\right), \Delta t\right)$$

via algebra.

But C is on $t' = 0$.

I.e. C has $t' = 0 \Leftrightarrow C = \Delta t\left[\left(\frac{v}{c}\right), c\right]$ for some Δt.

$C = (t, x)$, on $t' = 0 \Leftrightarrow -ct + \left(\frac{v}{c}\right)x = 0$

i.e. $t = \left(\frac{v}{c^2}\right) x$

It is handy to draw pictures with units $c=1$.