Last class:

$$
\begin{tikzcd}
 & E^2 \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\
\mathbb{R}^2 \arrow[rr, "\tau"'] & & \mathbb{R}^2
\end{tikzcd}
$$

E^2: the thing.
it abides

$\mathbb{R}^2$: your coordinates
your labels

ϕ_1, ϕ_2 invertible

τ: transition function

$\tau : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

my labels — your labels

$$\tau = \phi_2 \circ \phi_1^{-1}$$

Discussed preferred coordinates on E^2, with transition functions

$$\tau(x) = Hx + T \qquad H = \begin{bmatrix} c & \mp s \\ s & \pm c \end{bmatrix} \quad c^2+s^2=1$$

$$T \in \mathbb{R}^2$$

Euclidean transformations

In any of these coordinate systems $d(p_1,p_2)^2 = (p_2-p_1)^T(p_2-p_1)$

$$= \langle p_2-p_1, p_2-p_1 \rangle$$

$$= (p_2-p_1)\cdot(p_2-p_1)$$

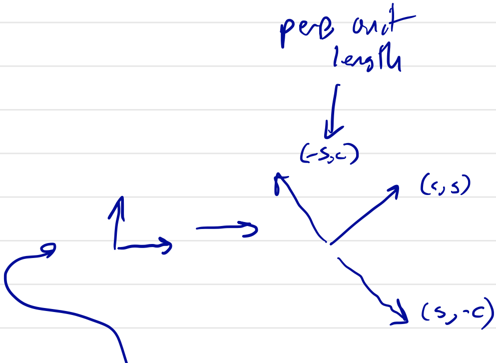
Notion of a group.

$GL(\mathbb{R}, n)$

$SL(\mathbb{R}, n)$

$O(n) : A^TA = I$

$SO(n)$



Exercise: $O(2) = \left\{ \begin{pmatrix} c & \mp s \\ s & \pm c \end{pmatrix} : s^2+c^2=1 \right\}$

$SO(2) = \left\{ \begin{pmatrix} c & -s \\ s & c \end{pmatrix} : s^2+c^2=1 \right\}$

E.g. X a set

$\mathrm{Sym}(X)$ is the set of all invertible functions $f: X \to X$

mult: function composition
inverses: function inverses.

In some sense, the groups $\mathrm{Sym}(X)$ are the granddaddies of them all. More interesting groups arise as subgroups of $\mathrm{Sym}(X)$ that preserve some extra structure on X.

Not to belabor a point, but an $n\times n$ matrix can be identified with a map from $\mathbb{R}^n \to \mathbb{R}^n$.

The matrix is A. The map is, say, f_A

$$(f_A(x))_i = \underset{\left(\sum_{j=1}^{n}\right)}{} A_{ij}\, x_j \qquad i = 1, \dots, n$$

Exercise: $f_{AB} = f_A \circ f_B$

I.e. $A \mapsto f_A$ is a group homomorphism $GL(\mathbb{R}, n) \to \mathrm{Sym}(\mathbb{R}^n)$

We'll blur this distinction.

Anyway, what structure is preserved by

a) $GL(\mathbb{R}, n)$? $0 \mapsto 0$, lines $\to$ lines] in fact, is characterized by this!

b) $SL(\mathbb{R}, n)$? above, plus: area $(\det = \pm 1)$; also, orientation TBA] ditto

c) $O(n)$? all of a), plus

Lengths of vectors:

$$\|Ax\|^2 = x^T A^T A x = x^T x = \|x\|^2.$$

distances between points

$$d(p_1, p_2)^2 = (p_2 - p_1)^T (p_2 - p_1)$$

$$\begin{aligned} d(Ap_1, Ap_2)^2 &= (Ap_2 - Ap_1)^T (Ap_2 - Ap_1) \\ &= (A(p_2 - p_1))^T A(p_2 - p_1) \\ &= (p_2 - p_1)^T A^T A (p_2 - p_1) \\ &= (p_2 - p_1)^T (p_2 - p_1) \\ &= d(p_1, p_2)^2. \end{aligned}$$

angle between x, y: $x \cdot y = |x|\,|y| \cos\theta$ (by def, in all dimensions)

This relies on Cauchy-Schwartz inequality:

$|x \cdot y| \le |x|\,|y|$ so $x \cdot y = |x|\,|y| \cdot \sigma$ for some $\sigma \in [-1, 1]$.

$$0 \le \| u - \lambda v \|^2$$
$$= \|u\|^2 - 2\lambda u \cdot v + \lambda^2 \|v\|^2$$

$\lambda = \dfrac{\langle u, v \rangle}{\|v\|^2}$ (min at this λ)

$$= \|u\|^2 - 2\frac{(u \cdot v)^2}{\|v\|^2} + \frac{(u \cdot v)^2}{\|v\|^2}$$

so $(u \cdot v)^2 \le \|u\|^2 \|v\|^2$

HW?

$$\langle Ax, Ay \rangle = x^T A^T A y$$
$$= x^T y$$
$$= \langle x, y \rangle$$

$\|Ax\|\,\|Ay\| \cos\theta_1$ $\qquad$ $\|x\|\,\|y\| \cos\theta_2$

$$\Rightarrow \theta_1 = \theta_2$$

$SO(n)$? $\det(A^TA) = \det(I) = 1$

$\quad \hookrightarrow = \det(A^T)\det(A)$

$\quad = \det(A)^2$

$\Rightarrow \det(A) = \pm 1.$

So only additional property is preservation of orientation.

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e.g. $X = \mathbb{R}^n$

structure: $d(p_1, p_2) = \sqrt{z^T z}$ $\quad z = p_2 - p_1$

isometry: $f: \mathbb{R}^n \to \mathbb{R}^n$, invertible,

$$d(f(p_1), f(p_2)) = d(p_1, p_2) \quad \forall p_1, p_2 \in \mathbb{R}$$

The set of isometries of $\mathbb{R}^n$ forms a group.

$\mathrm{Isom}(\mathbb{R}^2)$

What needs checking? $\quad f \circ h$ preserves dist

f^{-1} preserves dist if f does

Q: How are $\mathrm{Isom}(\mathbb{R}^n)$ and $O(n)$ related?

Clearly $O(n)$ is smaller. $0 \longmapsto 0$.

Fact (a little hard): $O(n)$ is the set of isometries of $\mathbb{R}^n$ that takes 0 to 0.

Exercise: Assuming this, show that every isometry of $\mathbb{R}^n$ has the form

$$f(x) = Hx + T \qquad H \in O(n)$$
$$T \in \mathbb{R}^n$$

Easier exercise: Show every map of above form is an isometry.

I.e. the Euclidean transformations are exactly the transformations that preserve distance on $\mathbb{R}^n$