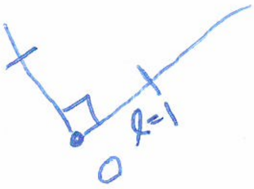
Symmetry Groups

A lot of this course will concern thinking about coordinates, and preferred coordinate systems.

E.g. We can label the Euclidean plane



$$\begin{tikzcd} (x,y) \arrow[r] \arrow[d] & \text{pt in plane } p(x,y) \\ \in \mathbb{R}^2 & \end{tikzcd}$$

(alt: $p \longmapsto (x,y)$)

A nice feature of this coordinate system

$\circ\ P_2 = (x_2, y_2)$

$\bullet\ P_1 = (x_1, y_1)$

$$d(P_1, P_2) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{Z^T \cdot Z} \qquad Z = P_2 - P_1$$

$$\begin{tikzcd}
 & \mathbb{E}^2 \arrow[dl, "\overset{\text{mine}}{\varphi_1}"'] \arrow[dr, "\overset{\text{yours}}{\varphi_2}"] & \\
\mathbb{R}^2 \arrow[rr, "\tau"'] & & \mathbb{R}^2
\end{tikzcd}$$

$$\tau \circ \varphi_1 = \varphi_2$$

$$\tau = \varphi_2 \circ \varphi_1^{-1}$$

$\downarrow$

transition from my labels to yours.

Important class of transition functions:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tau(x', y') = \underbrace{\begin{pmatrix} c & -s \\ s & c \end{pmatrix}}_{H} \begin{pmatrix} x' \\ y' \end{pmatrix} + \underbrace{\begin{pmatrix} t_x \\ t_y \end{pmatrix}}_{T}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = H \begin{pmatrix} x' \\ y' \end{pmatrix} + T \qquad c^2 + s^2 = 1$$

$$z = H z'$$

$$z^T z = z'^T \underbrace{H^T H}_{} z$$

$$\hookrightarrow \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{matrix} c^2+c2 & 0 \\ 0 & s^2+c^2 \end{matrix} = I$$

$$= z'^T z$$

$$d(p_1, p_2) = \begin{cases} \rightarrow \sqrt{z^T z} \\ \rightarrow \sqrt{(z')^T z'} \end{cases}$$

same formula in each coord system.

$$H_1(H_2 w + T_2) + T_1 = H_1 H_2 w + (H_1 T_2 + T_1)$$

$$H_2^T H_1^T H_1 H_2 = H_2^T I H_2 = H_2^T H_2 - I \quad ✓$$

Def: A map $\tau: \mathbb{R}^2 \to \mathbb{R}^2$ is called a Euclidean transformation if it has the form

$$\tau(x,y) = H \begin{bmatrix} x \\ y \end{bmatrix} + T$$

Where H is a 2×2 matrix, $T \in \mathbb{R}^2$, $H^T H = I$.

~~If in one coordinate system~~

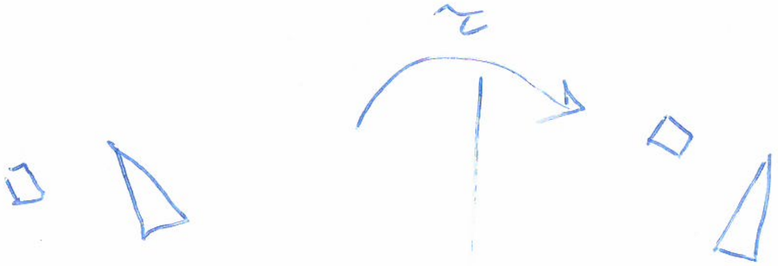
The distance formula is preserved under Euclidean transformation.

This is the "passive" perspective on Euclidean transformations.

There's an "active" perspective as well.

$$d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$d(P_1, P_2) = z^T z \qquad z = P_2 - P_1$$



$$d(\tau(p_1), \tau(p_2)) = w^T w \qquad w = \tau(p_2) - \tau(p_1)$$

$$= H(z) \qquad z = p_2 - p_1$$

$$= z^T z$$

$$= d(p_1, p_2)$$

It's a distance preserving map (a.k.a. an isometry).

The set of Euclidean transformations forms what is known as a 'group.'

$$G \times G \longrightarrow G$$

$$(a, b) \longrightarrow ab$$

with rules

1) $(ab)c = a(bc)$

2) There exists $1 \in G$, $1 \cdot g = g \cdot 1 = g$ $\forall g \in G$

3) For each $g \in G$ there exists h,

$$gh = hg = 1.$$

Exercise: a) If $1'g = g1'$ for all $g \in G$, $1' = 1$.
(we call 1 the group identity)

b) If $gh = hg = 1$

$$gh' = h'g = 1$$

Then $h = h'$.

We write $h = g^{-1}$, an call g^{-1} g's inverse, e.g. ① $\mathbb{R} \setminus \{0\}$ with multiplication

id: 1

$x^{-1} = \frac{1}{x}$

② $\mathbb{Z}$ under addition

identity : 0

$x^{-1} = -x$

③ 2x2 invertible matrices under matrix mult. → not commutative!

identity: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

inverse: matrix inverse

$GL(\mathbb{R}, 2)$ (general linear group)

$(GL(\mathbb{R}, n), GL(\mathbb{C}, n))$

4) $\mathbb{R}_+ \subseteq \mathbb{R} \setminus \{0\}$, same otherwise as ①

↳ $\{x \in \mathbb{R} : x > 0\}$

We call $\mathbb{R}_+$ a <u>subgroup</u> of $\mathbb{R} \setminus \{0\}$.

5) The 2x2 matrices with unit determinant,

$\det(AB) = \det(A)\det(B)$!

$SL(\mathbb{R}, 2) \subseteq GL(\mathbb{R}, 2)$ is a subgroup

6) $O(2) \subseteq GL(\mathbb{R}, 2)$

$$A^T A = I$$

known as the orthogonal group; preserves orthogonality of vectors:

i.e. if $x \cdot y = 0$ then
$$\begin{aligned}(Ax)\cdot(Ay) &= x^T A^T A y \\ &= x^T y \\ &= x \cdot y = 0.\end{aligned}$$

7) $SO(2) \subseteq O(2)$

$$\det(A) = 1$$

exercise: ~~If~~ H, K are subgroups of G, so is $H \cap K$.

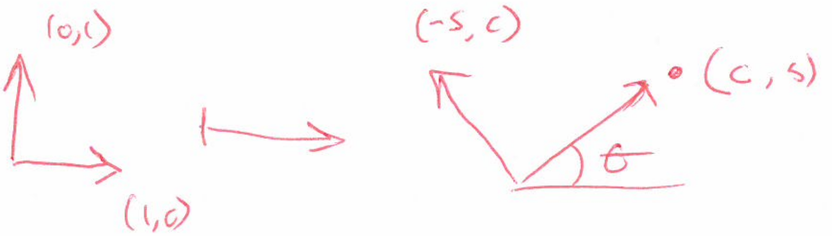
In fact, ~~$\det(A^TA)$~~
if $A \in O(2)$
$$\begin{aligned}1 &= \det(I) \\ &= \det(A^T A) \\ &= \det(A^T)\cdot\det(A) \\ &= \det(A)^2\end{aligned}$$

So $\det(A) = \pm 1$. Those in $SO(2)$ have $\det = 1$.

Exercise: If $A \in O(2)$ then

$$A = \begin{pmatrix} c & \mp s \\ s & \pm c \end{pmatrix}, \quad c^2+s^2=1,$$

& $A \in SO(2) \Rightarrow A = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$



$c = \cos\theta.$

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8) The set of all invertible maps $f: X \to X$. X some set

multiplication: function composition

id: identity map

inverses: function inverses, $\mathrm{Sym}(X)$

~~Lines thru [illegible] to~~

Frequently a group can be "thought of" as a collection of maps from a set to itself

$$G \xrightarrow{\Phi} \mathrm{Sym}(X)$$

(group hom)

$$\Phi(gh) = \Phi(g)\Phi(h)$$

G is identified with $\Phi(G)$, a

In particular, matrices determine functions

A an $n \times m$ matrix A yields a map $\mathbb{R}^m \to \mathbb{R}^n$, f_A.

$$x \longmapsto Ax$$

$$f_A(x)^i = A^i_j x^j$$

we'll blur the lines between the matrix and the function it represents,

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From this perspective, groups can often thought of as maps from a set to itself that preserve some extra structure on the set;

$GL(\mathbb{R},2)$: maps from $\mathbb{R}^2 \to \mathbb{R}^2$ that

a) set 0 to itself

b) take lines to lines.

$SL(\mathbb{R}_2)$: as above <u>and</u> preserve area and $\underbrace{\text{orientation}}$,

↳ more to follow,

O(2) maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ that

1) ~~preserve~~ take 0 to 0

2) preserve distance.

(its easy to see they all do this; harder to show this is all of them)

SO(2) as above and preserve orientation.

Exercise (main point for today).

The Euclidean group is a group.

In fact, it is the group of maps $\mathbb{R}^2 \to \mathbb{R}^2$ that preserves dis istance.

Dictionary

Isometry of space:

$f : \mathbb{R}^3 \to \mathbb{R}^3$

$f(x) = Hx + T$

$T \in \mathbb{R}^3, \quad H \in O(3) \qquad H^T H = I,$

proper: $H \in SO(3) \qquad \det(H) = 1.$

$$M \begin{pmatrix} t_1' \\ \vdots \\ 1 \end{pmatrix} + \begin{pmatrix} c_0 \\ \vdots \\ 1 \end{pmatrix} -$$

$$(1, 0, \dots, 0) M \begin{pmatrix} t_1' \cancel{)} - t_2' \\ \vdots \quad\quad \vdots \end{pmatrix} = \quad t_1' - t_2'$$

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ v_1 & & & \\ v_2 & & H & \\ v_3 & & & \end{array}$$