1. Suppose a particle travels with a force law of the form

$$\frac{dP}{d\tau} = AV$$

where P is its 4-momentum, V is its 4-velocity and A is a spacetime dependent matrix. Assuming that $g(P,P)$ does not depend on τ, show that

$$A = GB$$

where B is antisymmetric.

2. Let ω be a covector expressed in an inertial coordinate system. We define

$$(d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i. \tag{1}$$

a) Show that in a second inertial coordinate system $\hat{x} = Lx + Y$,

$$L^t \widehat{d\omega} L = d\omega. \tag{2}$$

Here,

$$(\widehat{d\omega})_{ij} = \hat{\partial}_i \hat{\omega}_j - \hat{\partial}_j \hat{\omega}_i. \tag{3}$$

b) If f is a function, show $d(df) = 0$.

c) Let $K = (K_1, K_2, K_3)$ and $L = (L_1, L_2, L_3)$ be triples of real numbers. We define the matrix

$$\mathcal{F}(K,L) = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & L_1 & -L_2 \\ -K_2 & -L_1 & 0 & L_3 \\ -K_3 & L_2 & -L_2 & 0 \end{pmatrix} \tag{4}$$

If $\omega = (\omega_0, \vec{\omega})$, show that

$$d\omega = \mathcal{F}(\partial_0 \vec{\omega} - \nabla \omega^0, \nabla \times \vec{\omega}) \tag{5}$$

Here we interpret $\vec{\omega}$ as the coordinates of a vector in $\mathbb{R}^3$ and ∇ and $\nabla\times$ are the standard gradient and curl operators in $\mathbb{R}^3$.

3. Recall that

$$\star\mathcal{F}(R,S) = \mathcal{F}(S,-R) \tag{6}$$

$$\star d\mathcal{F}(R,S) = [\operatorname{div} S, -\nabla \times R + \partial_0 S]. \tag{7}$$

a) Let ω be a 1-form. Show $\star dd\omega = 0$. (In fact, this shows $d^2 = 0$ acting on Λ^1)

b) Let F be a 2-form. Show $\delta\delta F = 0$.

c) Unwind the definitions and show that if ω is a 1-form, then

$$-\delta d\omega = \Box\omega - d\delta\omega. \tag{8}$$

4. Given a one-form ω we define the associated electric and magnetic fields E and B by

$$d\omega = \mathcal{F}(E, -cB).$$

Recalling that Maxwell's equations are

$$-\delta d\omega = \frac{1}{c\epsilon_0}(c\rho, -j)$$

show that E and B satisfy

$$\nabla \cdot E = \frac{1}{\epsilon_0}\rho \tag{9}$$

$$\frac{1}{c}\partial_0 E + \nabla \times B = \frac{1}{c^2\epsilon_0} j. \tag{10}$$

These are Gauss' Law and Ampere's equation respectively.

Then, from the fact that $\delta^2 = 0$ show that

$$\nabla \cdot B = 0 \tag{11}$$

$$c\partial_0 B + \nabla \times E = 0. \tag{12}$$

These are Gauss' Law for magnetism and Faraday's Law, respectively.

5. The fact that $d^2 = 0$ when acting on Λ^0 and Λ^1 has a partial converse. Use the results on page 184 of the text to prove the following.

a) Suppose on some ball that $d\omega = 0$ some some one-form ω. Show that there is a function f on the ball such that $\omega = df$.

b) Suppose on some ball that $\star dF = 0$ for some two-form F; this is equivalent to $dF = 0$ for the map $d : \Lambda^2 \to \Lambda^3$ that we did not discuss in detail. Show that there is a one-form ω with $F = d\omega$.