**1.** SR 7.2

**Solution:**
From problem 7.1, we have

$$(E')^2 = (m^2 + m^2)c^4 + 2Emc^2$$

where $E'$ is the rest energy of the $J/\psi$ particle and $E$ is the energy of the positron in the lab frame. But the rest energy of the $J/\psi$ particle is simply $Mc^2$ and hence

$$M^2c^4 = 2m^2c^4 + 2Emc^2$$

and

$$E = \frac{M^2c^2 - 2m^2c^2}{2m} = Mc^2\frac{M}{2m} - m^2c^2$$

The rest energy of the positron in $mc^2$ and hence the excess is

$$E - mc^2 = Mc^2\frac{M}{2m} - 2m^2c^2 = Mc^2\left[\frac{M}{2m} - \frac{2m}{M}\right].$$

Note that if instead we collide an electron and positron with opposite velocities $v$ and $-v$ the initial energy is

$$2mc^2\gamma(v) \tag{1}$$

and the final energy is

$$Mc^2 \tag{2}$$

as the resulting particle is at rest. Thus

$$2mc^2\gamma(v) = Mc^2 \tag{3}$$

The excess energy above the rest energy of the electron and the positron is then

$$Mc^2 - 2mc^2 = Mc^2\left[1 - \frac{2m}{M}\right]$$

This should be compared with our previous excess

$$Mc^2\left[\frac{M}{2m} - \frac{2m}{M}\right]$$

Since $m << M$, we conclude $M/(2m) > 1$ and thus the execess energy for the equal and opposite collision is less than that of the collision where the electron is sationary. The explanation for the difference is that in the first collision there is additional energy due to the velocity of the $J/\psi$ particle.

**2.** SR 7.3

**Solution, part a:**
In the observer's inertial frame we have the momenta

$$\begin{aligned} P_r &= m\gamma(u)(c,u) \\ P_e &= M\gamma(v)(c,-v) \\ P_r' &= m'\gamma(u')(c,u'). \end{aligned} \tag{4}$$

Here the subscript $r$ denotes the rocket, the subscript $e$ denotes the ejecta, and the prime denotes post-ejection. From conservation of 4 momentum we have

$$m\gamma(u) = M\gamma(v) + m'\gamma(u') \tag{5}$$

and

$$m\gamma(u)u = -M\gamma(v)v + m'\gamma(u')u'. \tag{6}$$

Using equation (5) to replace $m\gamma(u)$ in equation (6) we conclude

$$M\gamma(v)(u+v) + m'\gamma(u')(u-u') = 0. \tag{7}$$

Note also that since $P_r - P_e = P_r'$,

$$g(P_r - P_e, P_r - P_e) = g(P_r', P_r') = c^2(m')^2. \tag{8}$$

But we can directly compute

$$g(P_r - P_e, P_r - P_e) = g(P_r, P_r) + g(P_e, P_e) - 2g(P_r, P_e) = c^2m^2 + c^2M^2 - 2g(P_r, P_e).$$

Working in the rest frame of the rocket we see that $g(P_r, P_e) = \gamma(w)c^2 mM$. Hence

$$m^2 + M^2 - 2\gamma(w)mM = m'^2. \tag{9}$$

**Solution, part b:**
The equation

$$\gamma(v)\begin{pmatrix} c \\ -v \end{pmatrix} = \gamma(u)\gamma(w)\begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix}\begin{pmatrix} c \\ -w \end{pmatrix} \tag{10}$$

follows from the following facts:

- The rocket is traveling at velocity $u$ relative to the observer so the transformation from the rocket's frame to the observer's frame is

$$\gamma(u)\begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix}. \tag{11}$$

- The 4 velocity of the ejecta in the rocket's frame is $\gamma(w)(c,-w)$.

- The 4 velocity of the ejecta in the observer's frame is $\gamma(v)(c,-v)$.

**Solution, part c:**
Equation (10) can be equivalently written

$$\gamma(v)\gamma(u)\begin{pmatrix} 1 & -u/c \\ -u/c & 1 \end{pmatrix}\begin{pmatrix} c \\ -v \end{pmatrix} = \gamma(w)\begin{pmatrix} c \\ -w \end{pmatrix}. \tag{12}$$

Looking at the spatial part of this equation we conclude that

$$\gamma(v)\gamma(u)(u+v) = \gamma(w)w. \tag{13}$$

From equations (7) and (13)

$$mm'\gamma(u')\gamma(u)(u'-u) = mM\gamma(u)\gamma(v)(u+v) = mM\gamma(w)w. \tag{14}$$

Equation (9) then implies

$$mm'\gamma(u')\gamma(u)(u'-u) = \frac{1}{2}(m^2 + M^2 - (m')^2)w. \tag{15}$$

**Solution, part d:**
Setting $u' - u = \delta u$ and $m' - m = \delta m$ we find

$$mm'\gamma(u')\gamma(u)\delta u = \frac{1}{2}(-(m+m')\delta m + M^2)w \tag{16}$$

and hence

$$mm'\gamma(u')\gamma(u)\frac{\delta u}{\delta m} = -mw - \frac{1}{2}\delta m w + \frac{1}{2}\frac{M^2}{\delta m}w. \tag{17}$$

Using the fact that $m' \to m$ and $u' \to u$ as $\delta u, \delta m \to 0$ we find

$$m^2\gamma(u)^2\frac{du}{dm} = -mw \tag{18}$$

so long as $M^2/(\delta m) \to 0$. But equation (9) ensures that $\delta m$ and $M$ are linearly related in the limit as $M \to 0$, so indeed $M^2/(\delta m) \to 0$.

**Solution, part e:**
See text.

**3.** SR 3.3

**4.** SR 5.9 We note that

$$\sigma^2 = c^2t^2 - \sum (x^i)^2$$

and hence

$$\operatorname{Grad} \sigma^2 = 2(ct, -(-x^1), -(-x^2), -(-x^3)) = 2X.$$

Moreover, by the chain rule, for any functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^{1,3} \to \mathbb{R}$,

$$\operatorname{Grad} f \circ g(x) = f'(g(x))\operatorname{Grad} g.$$

Hence

$$\operatorname{Grad} f(\sigma^2) = 2f'(\sigma^2)X.$$

Finally, we observe that

$$\operatorname{Div} X = \partial_0 x^0 + \partial_1 x^1 + \cdots + \partial_3 x^3 = 4.$$

Now consider $f(x) = x^{-1}$ so

$$f(\sigma^2) = \frac{1}{g(X,X)}$$

Then

$$\operatorname{Grad} \frac{1}{g(X,X)} = -2\frac{1}{g(X,X)^2}X.$$

Now for any function $h(x)$,

$$\operatorname{Div}(h(x)X) = g(\operatorname{Grad} f, X) + f(x)\operatorname{Div} X$$

and hence

$$\operatorname{Div}\operatorname{Grad}\frac{1}{g(X,X)} = 4\frac{1}{g(X,X)^3}g(X,X) - \frac{1}{g(X,X)^2}\operatorname{Div} X = 4\left[\frac{1}{g(X,X)^2} - \frac{1}{g(X,X)^2}\right] = 0.$$

That is,

$$\frac{1}{ct^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}$$

solves the wave equation (off of the light cone).

**5.**

a) Let $\xi \in \mathbb{R}^3$, let $z \in \mathbb{C}$ and let $u(t,x) = ze^{i(\xi\cdot x - c|\xi|t))}$. Show that $u$ is a complex valued solution of the wave equation. Describe its real part as a wave. What is the speed of the wave? What direction is it travelling in? What is its frequency?

b) Let $\hat{f} : \mathbb{R}^3 \to \mathbb{C}$ be smooth and compactly supported (i.e., $f(\xi) = 0$ for $\xi$ outside of some large ball). Show that

$$U(t,x) = \int_{\mathbb{R}^3} \hat{f}(\xi) e^{i(\xi\cdot x - c|\xi|t))} d\xi$$

is a complex-valued solution of the wave equation (and hence its real and imaginary parts both solve the wave equation). How is the solution $U$ related to the kinds of solutions described in part a)?

c) Show that

$$u(t,x) = U(t,x) + U(-t,x) \tag{19}$$

solves the wave equation with $u_t(0,x) = 0$.

d) A result from Fourier analysis says that if $f : \mathbb{R}^3 \to \mathbb{C}$ is, say, continuous and

$$\int_{\mathbb{R}^3} |f|^2 \tag{20}$$

is finite, then

$$f(x) = \int_{\mathbb{R}^3} \hat{f}(\xi) e^{i\xi\cdot x} d\xi \tag{21}$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x) e^{-i\xi\cdot x} dx. \tag{22}$$

With this result in hand, describe a strategy for solving the initial value problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= 0 \\ u(0,x) &= \phi(x) \\ u_t(0,x) &= 0. \end{aligned}$$

e) Challenge: describe a strategy for solving the initial value problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= 0 \\ u(0,x) &= 0 \\ u_t(0,x) &= \psi. \end{aligned}$$

What new issues appear compared to part *d*?

**Solution:**

**Solution, part a:**
We can write $z = re^{i\theta}$ for some $r > 0$ and $\theta \in \mathbb{R}$. Then

$$u(t,x) = re^{i(\xi\cdot x - c|\xi|t + \theta)}.$$

Note that

$$u_t t = -c^2 |\xi|^2 u(t,x)$$

and

$$(\partial_i)^2 u = -(\xi^i)^2 u(t,x)$$

for $i = 1,2,3$. Hence

$$\Box u = \frac{1}{c^2} u_t t - \Delta u = (|\xi|^2 - |\xi|^2) u = 0.$$

So $u$ is a complex valued solution of the wave equation.

Its real part is

$$\Re u = r\cos(\xi \cdot x - c|\xi| t + \theta)$$

Let

$$f(s) = r\cos(|\xi|s + \theta).$$

Then

$$\Re u(t,x) = f(\mathrm{e}\cdot x - ct)$$

where $\mathrm{e} = \xi/|\xi|$. This exhibits $\Re u$ as a wave traveling in the direction e with velocity $c$. Regarding its frequency, consider a sationary observer starting at $x = 0$. This observer sees the function values

$$\Re u(t,0) = f(-ct) = r\cos(|\xi|ct + \theta).$$

Hence, in unit time, the argument to cos passes through $|\xi|c/(2\pi)$ periods. The frequency is therefore $|\xi|c/(2\pi)$.

**Solution, part b:**

Let

$$U(t,x) = \int_{\mathbb{R}^3} \hat{f}(\xi)e^{i(\xi\cdot x - c|\xi|t))}d\xi$$

That $U(t,x)$ solves the wave equation comes from linearity of the wave equation, together with the fact that we can commute integration and differentiation in this case. One always has to worry about switching limiting operations, and the hypotheses that $\hat{f}$ is smooth and compactly supported are sufficient (and are overkill). Note that $U$ is a superposition of waves with frequencies $c|\xi|/(2\pi)$ travelling in the directions $\xi/|\xi|$. The coefficeint $\hat{f}(\xi)$ describes the amplitude of the component wave with frequency $c|\xi|/(2\pi)$ and direction $\xi/|\xi|$.

**Solution, part c:**

Note that $(\partial_t)^2 U(-t,x) = U_{tt}(-t,x)$ and hence $\Box(U(-t,x)) = (\Box U)(-t,x) = 0$.

That $u(t,x) = U(t,x) + U(-t,x)$ solves the wave equation follows from linearity. Moreover

$$u_t(0,x) = U_t(0,x) - U_t(0,x) = 0.$$

That $\tilde{u}(t,x) = U(t,x) - U(-t,x)$ solves the wave equation also follows from linearity. Moreover

$$\tilde{u}(0,x) = U_t(0,x) - U_t(0,x) = 0.$$

**Solution, part d:**

Consider a solution of the form

$$u(t,x) = U(t,x) + U(-t,x)$$

where

$$U(t,x) = \int \hat{f}(\xi)e^{i(\xi\cdot x - c|\xi|t)}\,d\xi.$$

Note that

$$u(0,x) = \int 2\hat{f}(\xi)e^{i(\xi\cdot x}\,d\xi.$$

But we can write

$$\phi(x) = \int \hat{\phi}(\xi)e^{i(\xi\cdot x}\,d\xi$$

where

$$\hat{\phi}(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\phi(x)e^{-i\xi\cdot x}dx.$$

So we set $\hat{f}(\xi) = \hat{\phi}(\xi)/2$. Strictly speaking, this is a strategy, not a solution. In part b, in order to allow us to glibly interchange integration and differentiation, I added the hypothesis that $\hat{f}$ was smooth and compactly supported. This need not be true for $\phi$. Nevertheless, when $\phi$ is square integrable, it turns out that this strategy can be made rigorous.

**Solution, part e:**

Now consider a solution of the form

$$u(t,x) = U(t,x) + U(-t,x)$$

where

$$U(t,x) = \int \hat{f}(\xi)e^{i(\xi\cdot x - c|\xi|t)}\,d\xi.$$

Note that

$$u_t(0,x) = \int -2c|\xi|i\hat{f}(\xi)e^{i(\xi\cdot x}\,d\xi,$$

at least when $\hat{f}$ is compactly supported. We would like

$$u_t(0,x) = \psi(x) = \int \hat{\psi}(\xi)e^{i(\xi\cdot x}\,d\xi$$

where

$$\hat{\psi}(\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\psi(x)e^{-i\xi\cdot x}dx.$$

So we require

$$\hat{f}(\xi) = -\frac{1}{2c|\xi|}\hat{g}(\xi).$$

Again, we have all the caveats as before: $\hat{f}$ need not be smooth, nor need it be compactly supported. The division by $|\xi|$ is a new, troublesome wrinkle, and to make this a formal solution, one would need to also ensure that the singularity in $\hat{g}$ at 0 does not pose a problem.