1. SR 7.2

2. SR 7.3

3. SR 3.3

4. SR 5.9

5.

a) Let $\xi \in \mathbb{R}^3$, let $z \in \mathbb{C}$ and let $u(t,x) = ze^{i(\xi \cdot x - c|\xi|t))}$. Show that u is a complex valued solution of the wave equation. Describe its real part as a wave. What is the speed of the wave? What direction is it travelling in? What is its frequency?

b) Let $\hat{f} : \mathbb{R}^3 \to \mathbb{C}$ be smooth and compactly supported (i.e., $f(\xi) = 0$ for ξ outside of some large ball). Show that

$$U(t,x) = \int_{\mathbb{R}^3} \hat{f}(\xi) e^{i(\xi \cdot x - c|\xi|t))} d\xi$$

is a complex-valued solution of the wave equation (and hence its real and imaginary parts both solve the wave equation). How is the solution U related to the kinds of solutions described in part a)?

c) Show that

$$u(t,x) = U(t,x) + U(-t,x) \tag{1}$$

solves the wave equation with $u_t(0,x) = 0$.

d) A result from Fourier analysis says that if $f : \mathbb{R}^3 \to \mathbb{C}$ is, say, continuous and

$$\int_{\mathbb{R}^3} |f|^2 \tag{2}$$

is finite, then

$$f(x) = \int_{\mathbb{R}^3} \hat{f}(\xi) e^{i\xi \cdot x} d\xi \tag{3}$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x) e^{-i\xi \cdot x} dx. \tag{4}$$

With this result in hand, describe a strategy for solving the initial value problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= 0 \\ u(0,x) &= \phi(x) \\ u_t(0,x) &= 0. \end{aligned}$$

e) Challenge: describe a strategy for solving the initial value problem

$$\begin{aligned} u_{tt} - c^2 \Delta u &= 0 \\ u(0,x) &= 0 \\ u_t(0,x) &= \psi. \end{aligned}$$

What new issues appear compared to part *d*?