1. SR 6.3

Solution:
The vector from event A to B has direction U and interval σ. Since U has interval c,

$$B = A + \frac{\sigma}{c} U.$$

Similarly, $C = B + \frac{\tau}{c} V$ and $C = A + \frac{\tau'}{c} V'$. Thus

$$A + \frac{\tau'}{c} V' = A + \frac{\sigma}{c} U + \frac{\tau}{c} V.$$

Subtracting A and multiplying by c obtains the relation

$$\tau' V' = \sigma U + \tau V.$$

But then

$$\tau'^2 c^2 = g(\tau' V', \tau' V') = g(\sigma U + \tau V, \sigma U + \tau V) = \sigma^2 c^2 + \tau^2 c^2 + 2\sigma\tau g(U, V).$$

In the frame in which the traveler is at rest in the first part of his journey, $U = (c, 0)$ and $V = \gamma(v)(c, v)$ so $g(U, V) = c^2\gamma(v)$. Thus

$$\tau'^2 = sigma^2 + \tau^2 + 2\sigma\tau\gamma(v).$$

Since $v \neq 0$, $\gamma(v) > 1$. And since $\tau, \sigma > 0$,

$$\tau'^2 = sigma^2 + \tau^2 + 2\sigma\tau\gamma(v) > sigma^2 + \tau^2 + 2\sigma\tau = (\sigma + \tau)^2.$$

We conclude that $\tau' > \sigma + \tau$.

Of course, in classical mechanics, the time difference between the two paths is identical, $\tau' = \sigma + \tau$.

The interesting phenomenon here is that the **longest** path from A to C is the one for the non-accelerating traveler.

2. SR 6.4

Solution:
We may assume the traveler is traveling in the t, x plane and we will ignore the other directions. So $\alpha(\tau) = (ct(\tau), x(\tau))$ is its path parameterized by proper time. Now α' always has length c; this is what it means to be parameterized by proper time. So for each τ there is a uniqe rapidity $\psi(\tau)$ such that the 4-velocity

$$V = \alpha'(\tau) = c(\cosh(\psi), \sinh(\psi)).$$

Taking another derivative with respect to τ,

$$A = \alpha''(\tau) = c(\sinh(\psi), \cosh(\psi))\frac{d\psi}{d\tau}$$

Since $(\sinh(\psi), \cosh(\psi))$ is spacelike with interval −1,

$$g(A,A) - c^2\left(\frac{d\psi}{d\tau}\right)^2$$

3. SR 6.5

Solution:
Let

$$\alpha_A(\tau) = (c^2/a)[\sinh(a\tau/c), -\cosh(a\tau/c)] \tag{1}$$
$$\alpha_B(\tau) = (c^2/a)[\sinh(a\tau/c), \cosh(a\tau/c)] \tag{2}$$

be the paths of the two rockets that are accelerating in opposite directions with acceleration a. Then

$$Z(\tau) = \alpha_B(\tau) - \alpha_A(-\tau) = \frac{2c^2}{a}[\sinh(a\tau/c), \cosh(a\tau/c)].$$

Thus $g(Z,Z) = -4c^4/a^2$ for all τ. Moreover,

$$\alpha_A'(-\tau) = c[\cosh(-a\tau/c), -\sinh(-a\tau/c)] = c[\cosh(a\tau/c), \sinh(a\tau/c)] \tag{3}$$
$$\alpha_B'(\tau) = c[\cosh(a\tau/c), \sinh(a\tau/c)] \tag{4}$$

which are identical. And

$$g(\alpha_A'(-\tau), Z(\tau)) = \frac{2c^3}{a}\left[\sinh(a\tau/c)\cosh(a\tau/c) - \sinh(a\tau/c)\cosh(a\tau/c)\right] = 0.$$

So Z is a spacelike vector orthogonal to $\alpha_A'(-\tau)$ and therefore rocket A sees the event $\alpha_A(-\tau)$ as simultaneous with the event $\alpha_B(\tau)$, at a constant distance $2c^2/a$. The same holds for rocket B since $\alpha_B'(\tau) = \alpha_A'(-\tau)$.

This is a difficult problem to reconcile with our intuition from classical mechanics. The key observation is that as rocket A travels into the future, the parts of B's worldline that A deduces are 'now' actually travel back into the past. So even though A and B are accelerating in opposite directions, in effect A is seeing the course of time for B run in reverse. At each moment the velocities of A and B are deemed equal by A and thus it is no surprise to find that the distance between the rockets remains the same.

Sketching the world lines of A and B it's easy to see, however, that light from rocket A will never reach rocket B. So although B is a constant distance from A, according to A, the rocket A never knows this.

4. Let $\kappa(s)$ be a function on $\mathbb{R}$ and let

$$\phi(s) = \frac{1}{c}\int_0^s \kappa(r)\,dr. \tag{5}$$

Show that

$$\alpha(s) = c\int_0^s (\cosh(\phi(s)), \sinh(\phi(s))\,ds \tag{6}$$

is pararameterized by proper time and has a 4-acceleration with size $|\kappa(s)|$. What does the sign of κ tell you?

Solution:
From the Fundamental Theorem of Calculus,

$$\phi'(s) = \frac{1}{c}\kappa(s)$$

and

$$\alpha'(s) = c(\cosh(\phi(s)), \sinh(\phi(s)))$$

Evidently $g(\alpha', \alpha') = c^2$ and thus α is parameterized by proper time. Moreover,

$$\alpha''(s) = c(\sinh(\phi(s)), \cosh(\phi(s)))\phi'(s) = (\sinh(\phi(s)), \cosh(\phi(s)))\kappa(s).$$

Therefore $g(\alpha', \alpha') = -\kappa^2$ and curve has acceleration of size κ. Since the x-component of α'' is $\cosh(\phi(s))\kappa(s)$, and since $\cosh(\phi) > 0$, the sign of κ determines if the acceleration is to the left or to the right.

5. Using some kind of computer technology, generate a graph of a curve in spacetime with acceleration

$$\kappa(s) = \sin(s) \tag{7}$$

over the interval $s \in [0, 2\pi]$.

Solution:
We use units in which $c = 1$.

Following the recipe from the previous problem, if $\kappa(s) = \sin(s)$ we can take

$$\phi(s) = \frac{1}{c}\int_0^s \sin(r)\, dr = 1 - \cos(s)$$

But for purposes of our diagram, it's more convenient to take $\phi(s) = -\cos(s)$, which does not change $\phi'(s)$ and simply corresponds to an overall boost. Then

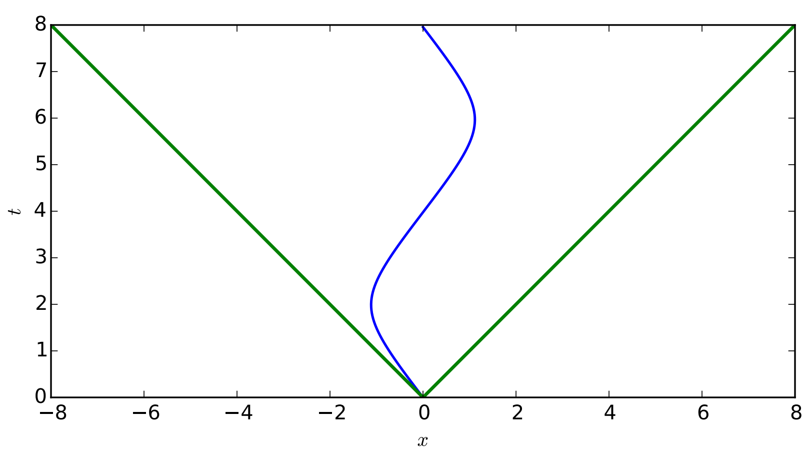
$$\alpha(s) = \int_0^s (\cosh(\cos(s)), -\sinh(\cos(s))\, ds$$

This integration can be computed numerically, e.g. python's `scipy.integrate.quad`.

Here's my code

```
from math import *
from scipy.integrate import quad
import numpy as np
import matplotlib.pyplot as pp

def x(s):
return quad( lambda z: sinh(-cos(z)),0,s)
```

```
def t(s):
return quad(lambda z: cosh(-cos(z)),0,s)

tau = np.linspace(0,2*pi,100)

X = [x(s) for s in tau]
T = [t(s) for s in tau]

pp.plot(X,T,color='blue',linewidth=1.5)
tmax = np.max(T)
pp.plot([0,tmax],[0,tmax],color='green',linewidth=2)
pp.plot([0,-tmax],[0,tmax],color='green',linewidth=2)
pp.axis('scaled')
pp.xlabel('$x$')
pp.ylabel('$t$')
pp.savefig('HW5_f1.pdf')
pp.show()
```



6. SR 7.1

Solution:
Before the collision, we have momenta P_1 and P_2 with $g(P_1, P_1) = M^2c^2$ and $g(P_2, P_2) = m^2c^2$. After the collision we have momentum $P = P_1 + P_2$. Moreover, $c^2 g(P, P)$ is the square of the rest energy of a particle with momentum P: after all, in its rest frame $P =$

$(cm_0, 0)$ and $c^2 g(P,P) = c^2 c^2 m_0^2 = (m_0 c^2)^2$ as desired. Thus

$$(E')^2 = c^2 g(P,P) = c^2 g(P_1 + P_2, P_2 + P_2) = c^2 g(P_1, P_1) + c^2 g(P_2, P_2) + 2c^2 g(P_1, P_2).$$

Now $g(P_1, P_1) = c^2 M^2$ and $g(P_2, P_2) = c^2 m^2$. Moreover, since P_1 has energy E in the lab fame,

$$P_1 = (E/c, *).$$

Sicne $P_2 = (cm, 0)$ in the lab frame, $g(P_1, P_2) = (E/c)cm = Em$. Thus

$$(E')^2 = c^2(c^2 M^2 + c^2 m^2 + 2Em)$$

as desired.