

1. SR 5.4

Let O and O' be two non-accelerating observers with inertial coordinate systems related by a proper orthochronous Lorentz transformation L , with $E = LE'$. Show that

$$L = \begin{pmatrix} \gamma & -\gamma u'_1/c & -\gamma u'_2/c & -\gamma u'_3/c \\ \gamma u_1/c & * & * & * \\ \gamma u_2/c & * & * & * \\ \gamma u_3/c & * & * & * \end{pmatrix} \quad (1)$$

Solution:

The position of the origin in the O' is given by $(t', 0, 0, 0)$. Sending these events through L we find

$$(ct \ x \ y \ z) = ct' (L_0^0 \ L_0^1 \ L_0^2 \ L_0^3). \quad (2)$$

Hence

$$t = L_0^0 t' \quad (3)$$

and

$$u_1 = \frac{dx}{dt} = \frac{1}{L_0^0} \frac{dx}{dt'} = c \frac{L_0^1}{L_0^0}. \quad (4)$$

Thus

$$L_0^1 = L_0^0 \frac{u_1}{c}. \quad (5)$$

A similar equation holds for y and z . Letting γ denote the value of L_0^0 we find

$$L = \begin{pmatrix} \gamma & a_1 & a_2 & a_3 \\ \gamma u_1/c & * & * & * \\ \gamma u_2/c & * & * & * \\ \gamma u_3/c & * & * & * \end{pmatrix} \quad (6)$$

for some numbers γ , a_1 , a_2 , and a_3 . But

$$L^t GL = G \quad (7)$$

and consequently from the upper-left entry of this matrix equation

$$\gamma^2 (1 - (u_1/c)^2 + (u_2/c)^2 + (u_3/c)^2) = 1. \quad (8)$$

Since L is orthochronous, $\gamma > 0$ and hence

$$\gamma = \frac{1}{\sqrt{1 - [(u_1)^2 + (u_2)^2 + (u_3)^2]/c^2}} = \gamma(u) \quad (9)$$

where $u = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

Similar considerations applied to L^{-1} show that L^{-1} has the form

$$L^{-1} = \begin{pmatrix} \gamma' & * & * & * \\ \gamma' u'_1/c & * & * & * \\ \gamma' u'_2/c & * & * & * \\ \gamma' u'_3/c & * & * & * \end{pmatrix} \quad (10)$$

where $\gamma' = \gamma(u')$ and where $u' = \sqrt{(u'_1)^2 + (u'_2)^2 + (u'_3)^2}$. But

$$L^t = GL^{-1}G = \begin{pmatrix} \gamma' & * & * & * \\ -\gamma' u'_1/c & * & * & * \\ -\gamma' u'_2/c & * & * & * \\ -\gamma' u'_3/c & * & * & * \end{pmatrix} \quad (11)$$

But

$$L^t = \begin{pmatrix} \gamma & \gamma u_1/c & \gamma u_2/c & \gamma u_3/c \\ a_1 & * & * & * \\ a_2 & * & * & * \\ a_3 & * & * & * \end{pmatrix} \quad (12)$$

from which we conclude that $\gamma' = \gamma$ and $a_i = -\gamma u'_i/c$ as required.

2. SR 5.5

Solution:

Let the matrices from the problem be L_1 , L_2 , L_3 and L_4 respectively. It is an easy computation to show that

$$L_1^t G L_1 = G$$

and that L_1 has unit determinant. Since its upper-left entry is positive, this is a proper, orthochronous Lorentz transformation.

It is a tedious computation (use Matlab/python/Julia/anything!) to check that

$$L_2^t G L_2 = G$$

and that $\det(L_2) = -1$. Since its upper left entry is positive this is an orthochronous but not proper Lorentz transformation.

Notice that L_3 is obtained from L_2 by multiplying the first column by -1 and interchanging the middle columns. Multiplying the first column by -1 is effected by right multiplication by

$$\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$$

which is evidently a Lorentz transformation. Interchanging the middle columns is effected by right multiplication by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is readily seen to be a Lorentz transformation (it's a spatial reflection). Hence the composition of L_2 with these two matrices is also a Lorentz transformation. Its determinant is the same as L_2 since the sign changes for the column interchange and the first

column multiplication cancel. Since the upper-left entry is negative, we find L_3 is a non-proper, non-orthochronous Lorentz transformation.

By these same arguments, L_4 is a proper, orthochronous Lorentz transformation. Its determinant has the opposite sign of that of L_3 , and its upper-left entry is positive.

3. SR 5.6

Let V be a four-vector. Show

- i) If V is future pointing timelike, there is an inertial coordinate system in which it has components $(a, 0, 0, 0)$ with $a = \sqrt{g(V, V)}$.
- ii) If V is future pointing null, there is an inertial coordinate system in which it has components $(1, 1, 0, 0)$ with $a = \sqrt{g(V, V)}$.

Solution:

For part i), assume that V is future pointing and timelike. After applying a spatial rotation we can assume that $V = (V^0, V^1, 0, 0)$. Pick ϕ by

$$\tanh(\phi) = -\frac{V^1}{V^0},$$

which is possible since \tanh is invertible and $V^0 > 0$. Setting $C = \cosh(\phi)$ and $S = \sinh(\phi)$, we define

$$\begin{pmatrix} \hat{V}^0 \\ \hat{V}^1 \end{pmatrix} \begin{pmatrix} C & S \\ S & C \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \end{pmatrix} = \begin{pmatrix} CV^0 + SV^1 \\ SV^0 + CV^1 \end{pmatrix}$$

But

$$\hat{V}^1 SV^0 + CV^1 = \frac{C}{V^0} \left(-\tanh(\phi) + \frac{V^1}{V^0} \right).$$

Since spatial rotations and the above boost are orthochronous, so are their composition and we find that we can take V to a vector $\hat{V} = (a, 0, 0, 0)$ by an orthochronous Lorentz transformation. Since V is future pointing, so is \hat{V} and $a > 0$. Moreover,

$$\sqrt{g(V, V)} = \sqrt{g(LV, LV)} = \sqrt{g(aT, aT)} = a\sqrt{g(T, T)} = a. \quad (13)$$

Part ii). Let N be a null vector. Just as in our proof in class of part i), we can find an inertial frame where N has coordinates $N^1 \geq 0$ and $N^2 = N^3 = 0$. Moreover, since N is null,

$$(N^0)^2 - (N^1)^2 = 0 \quad (14)$$

and since $N^0 > 0$ (N is future pointing) and since $N^1 > 0$ we conclude

$$N = (a, a, 0, 0) \quad (15)$$

for some $a > 0$.

We now show by applying a boost that we can transform to $(1, 1, 0, 0)$. Note that for any rapidity ϕ ,

$$\begin{pmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{pmatrix} \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} e^\phi \\ e^\phi \end{pmatrix}. \quad (16)$$

Letting $\phi = -\log(a)$ we find

$$a(e^\phi, e^\phi) = (1, 1) \quad (17)$$

as needed. Hence we can boost in the (t, x) plane by rapidity ϕ to transform N to $(1, 1, 0, 0)$.

4. SR 5.7

Show that

- i) The sum of future pointing timelike vectors is future pointing timelike.
- ii) The sum of future pointing null vectors is future pointing timelike or future pointing null, and is null if and only if the vectors are linearly dependent.
- iii) Every four vector orthogonal to a timelike vector is spacelike.

Solution:

Lemma 4: Let X and Y be causal and future pointing. Then

$$g(X, Y) \geq \|X\| \|Y\| \quad (18)$$

with equality if and only if X and Y are linearly dependent.

Proof. First, suppose X is timelike. Without loss of generality, we may assume

$$X = (X^0, 0, 0, 0) \quad (19)$$

for some $X > 0$. Then

$$g(X, Y) = X^0 Y^0 = \|X\| \|Y\| \quad (20)$$

Note that

$$\|Y\| = \sqrt{(Y^0)^2 - (Y^1)^2 - (Y^2)^2 - (Y^3)^2} \leq |Y^0| = Y^0 \quad (21)$$

with equality if and only if $Y^1 = Y^2 = Y^3 = 0$ (i.e. if and only if Y and X are linearly dependent. Hence

$$g(X, Y) \geq \|X\| \|Y\| \quad (22)$$

with equality if and only if X and Y are linearly dependent.

The case where Y is timelike follows from our previous argument and symmetry of g , so we now suppose both X and Y are null. Since $\|X\| = \|Y\| = 0$, we need to show that

$$g(X, Y) \geq 0 \quad (23)$$

with equality if and only if X and Y are linearly dependent. Without loss of generality we may assume

$$X = (1, 1, 0, 0) \quad (24)$$

and hence

$$g(X, Y) = Y^0 - Y^1. \quad (25)$$

Since Y is null and future pointing

$$Y^0 = \sqrt{(Y^1)^2 + (Y^2)^2 + (Y^3)^2}. \quad (26)$$

and hence

$$Y^0 \geq |Y^1| \quad (27)$$

with equality if and only if $Y^2 = Y^3 = 0$. Now

$$Y^0 - Y^1 \geq |Y^1| - Y^1 \geq 0 \quad (28)$$

with equality if and only if $Y^2 = Y^3 = 0$ and $Y^1 \geq 0$.

Notice $Y = (Y^0, Y^1, 0, 0)$ with $Y^1 > 0$ is equivalent to $Y = Y^0(1, 1, 0, 0)$ since Y is null and future pointing. Hence in the null case we find

$$g(X, Y) \geq 0 \quad (29)$$

with equality if and only if $Y = Y^0 X$ as needed. \square

Now with the main solution.

For parts i) and ii), suppose X and Y are causal and future pointing. Expressing the vectors with respect to an inertial frame, $(X + Y)^0 = X^0 + Y^0 > 0$ since X^0 and Y^0 are positive. Hence $X + Y$ is future pointing.

Lemma 4 implies

$$\begin{aligned} g(X + Y, X + Y) &= g(X, X) + g(Y, Y) + 2g(X, Y) \\ &\geq g(X, X) + g(Y, Y) + 2\|X\|\|Y\| \\ &= (\|X\| + \|Y\|)^2 \\ &\geq 0 \end{aligned} \quad (30)$$

with equality if and only if X and Y are linearly dependent and both null. Hence Z is timelike unless X and Y are null and linearly dependent, in which case Z is null.

For part iii), suppose X is timelike and $g(X, Y) = 0$. From our extended version of part i) we know that if Y is causal and future pointing that $g(X, Y) > 0$. If Y is causal and past pointing, then $-Y$ is causal and future pointing and $g(X, -Y) > 0$ and $g(X, Y) < 0$. Hence if $g(X, Y) = 0$, then Y is neither causal future pointing, nor causal past pointing. That leaves spacelike!

5. SR 5.8 Let X and Y be future pointing and let $Z = X + Y$. Then

$$\sqrt{g(Z, Z)} \geq \sqrt{g(X, X)} + \sqrt{g(Y, Y)} \quad (31)$$

with equality if and only if X and Y are null and linearly dependent.

Solution:

Lemma 4 implies

$$(\|X\| + \|Y\|)^2 = \|X\|^2 + \|Y\|^2 + 2\|X\|\|Y\| \leq \|X\|^2 + \|Y\|^2 + 2g(X, Y) = g(X + Y, X + Y) = \|Z\|^2 \quad (32)$$

with equality if and only if X and Y are linearly dependent. Hence

$$\|Z\| \geq \|X\| + \|Y\| \quad (33)$$

with equality if and only if X and Y are linearly dependent.

This result is akin to the triangle inequality for Euclidean space:

$$\|X + Y\| \leq \|X\| + \|Y\| \quad (34)$$

with equality if and only if X and Y are linearly dependent. The inequality in the Euclidean case points in the opposite direction.

6. [The Peter Mulvey Observation] Your head will be younger than your feet by the time of your death. Estimate, with justification, how much younger.

Solution:

We apply equation (6.14) from the text,

$$\frac{d\tau}{dt} = \sqrt{1 - R^2\omega^2/c^2} \quad (35)$$

concerning the relationship between inertial time and proper time for circular motion. The radii in question are the radius of the earth,

$$R_0 = 6400\text{km} \quad (36)$$

and radius, $R_1 = R_0 + 2/1000$ corresponding to one's head. We then have

$$\frac{d\tau_1}{dt} - \frac{d\tau_0}{dt} \approx -\frac{1}{1 - R_0^2\omega^2/c^2} R_0(R_1 - R_0) \frac{\omega^2}{c^2} = 7.5 \times 10^{-19} \text{ seconds.} \quad (37)$$

Taking the elapsed t to be 100 years we find

$$\tau_1 - \tau_0 = 2.4 \times 10^{-9}, \quad (38)$$

i.e about 3 nanoseconds. Of course, one is not standing one's whole life, and one is not always standing on the equator of the earth, so this estimate is an overestimate. Still, the net discrepancy on the order of one nanosecond, the about the time it takes my laptop to execute a single instruction.