1. SR 4.3

See text, page 176.

2. SR 4.4

Two light sources are at rest distance D apart and emit photons in the positive x direction. Show that in a frame where the sources have velocity u along the x-axis, the photons have distance

$$D\sqrt{\frac{c-u}{c+u}}$$

apart.

Solution:
Let the coordinates of the moving frame be denoted by $(\hat{t}, \hat{x})$ so we can take

$$\begin{pmatrix} c\hat{t} \\ \hat{x} \end{pmatrix} = \gamma \begin{pmatrix} 1 & u/c \\ u/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Let L denote this boost.

We will assume that the emissions occur at $t = 0$ and that source S_1 is at $x = 0$ and source S_2 is at $x = D$. In the frame where the sources are moving the emission from S^1 is at $(0, 0)$ again but the emission from S^2 has coordinates

$$L \begin{pmatrix} 0 \\ D \end{pmatrix} = \gamma D \begin{pmatrix} u/c \\ 1 \end{pmatrix}.$$

Note that the emission from S_2 occured after the emission at S_1 in the moving frame at time $c\hat{t} = \gamma D(u/c)$. At this time, the photon from S_1 will be at position $c\hat{t} = \gamma D(u/c)$ and hence

$$\Delta\hat{x} = \gamma D - \gamma D(u/c) = D\frac{1-(u/c)}{\sqrt{1-(u/c)^2}} = D\sqrt{\frac{c-u}{c+u}}$$

This distance remains constant since the photons are both travelling to the right with speed c.

3. The interval between events $E_1 = (t_1, x_1)$ and $E_2 = (t_2, x_2)$ in some inertial coordinate system is

$$c^2(t_1 - t_2)^2 - (x_1 - x_2)^2. \tag{1}$$

Suppose $\iota : \mathbb{R}^2 \to \mathbb{R}^2$ is a transformation that preserves the interval between any two events. Assuming that ι is affine, show that there is a (possibly non-proper or non-orthochronus) Lorentz transformation L and a vector $b \in \mathbb{R}^2$ such that

$$\iota(E) = \mathcal{C}^{-1}L\mathcal{C}E + b. \tag{2}$$

Here $\mathcal{C}$ is the 2×2 diagonal matrix with diagonal entries c and 1. Hint: This problem should feel very familiar! And take advantage of problem 4.2!

Solution:
Suppose ι is affine and preserves the interval. Suppose first that ι takes the origin to the origin, so ι is linear. So we can write

$$\iota(E) = C^{-1}LCE \tag{3}$$

for some matrix L.

Let E be an event. Then

$$\mathrm{Int}(0, E) = E^t C^t G C E \tag{4}$$

and

$$\mathrm{Int}(\iota(0), \iota(E)) = \mathrm{Int}(0, \iota(E)) = \mathrm{Int}(0, C^{-1}LCE) = E^t C^t L^t G L C E. \tag{5}$$

So for all events E,

$$E^t C^t G C E = E^t C^t L^t G L C E. \tag{6}$$

Let B be the symmetric bilinear form

$$B(E, F) = E^t C^t G C F \tag{7}$$

and let $\hat{B}$ be the symmetric bilinear form

$$\hat{B}(E, F) = E^t C^t L^t G L C F. \tag{8}$$

Since B and $\hat{B}$ agree on the diagonal they are the same, and we conclude that

$$C^t G C = C^t L^t G L C. \tag{9}$$

Multiplying on the right by C^{-1} and on the left by $(C^t)^{-1} = C^{-1}$ we conclude that

$$G = L^t G L. \tag{10}$$

Since $G^0{}_0 = 1$ we conclude that

$$(L^0{}_0)^2 - (L^0{}_1)^2 - (L^0{}_2)^2 - (L^0{}_3)^2 = 1 \tag{11}$$

and consequently $L^0{}_0 \neq 0$. Moreover, since $\det G \neq 0$ we know that $\det L \neq 0$. Thus there are four possibilities for the sign of $L^0{}_0$ and $\det L$ (e.g., both positive, both negative, etc.). If $L^0{}_0 > 0$ and $\det L > 0$ then we know from problem 4.2 that L is a proper, orthochronus Lorentz transformation. On the other hand, let

$$F = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

It is easy to see that $F^t G F = G$ and hence

$$(FL)^t G F L = L^t F^t G F L = L^t G L = G. \tag{13}$$

Similarly $(RL)^t G R L = G$ and $(RFL)^t G (RFL) = G$. It is easy to see that one of L, RL, FL, and RFL has positive determinant and a positive upper-left entry. Hence one of these is a proper orthochronus Lorentz transformation and L is therefore the composition of a proper orthochronus Lorentz transformation with a space or time reflection (or both).

4. For simplicity the following problem is to be done in one space dimension. Suppose in the frame of some inertial observer a function has the form

$$f(t,x) = \sin(\omega t) \tag{14}$$

for some angular frequency ω. Now consider the frame of some observer traveling with velocity v relative to the original frame. Determine the time difference between peaks of the function as seen by the boosted observer.

Solution:
Let $(\hat{t}, \hat{x})$ be inertial coordinates for the boosted observer. Then

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} c\hat{t} \\ \hat{x} \end{pmatrix}. \tag{15}$$

In particular,

$$t = \gamma(v) \left[\hat{t} + (v/c^2)\hat{x} \right]. \tag{16}$$

Thus the function for the boosted observer is

$$f(\hat{t}, \hat{x}) = \sin(\omega\gamma(v)(\hat{t} + (v/c^2)\hat{x})). \tag{17}$$

Our observer has coordinates with $\hat{x} = 0$ and thus at time $\hat{t}$, the observer measures the function values to be

$$\sin(\omega\gamma(v)\hat{t}). \tag{18}$$

The observed period is therefore

$$\frac{2\pi}{\omega\gamma(v)}. \tag{19}$$

5. Pions are subatomic particles with a half life of $\Delta t = 1.8 \times 10^{-8}$ seconds. As a consequence, given a collection of pions left alone for a time Δt, half of the pions will decay into other particles.

A beam of pions is traveling at a speed $v = 0.99c$. Notice that in time Δt the beam travels

$$\Delta x = 0.99c\Delta t = 5.35\text{m} \tag{20}$$

and one might expect that the beam diminishes in intensity by one half every 5.35m. Instead, it deminishes by one half every 38m or so. Explain the discrepancy.

Solution:
Let $(\hat{t}, \hat{x})$ be coordinates in the rest frame of the pions, and let (t, x) be coordinates in the rest frame of the lab. So

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} c\hat{t} \\ \hat{x} \end{pmatrix}. \tag{21}$$

where $v/c = 0.99$. In the rest frame of the pions, the beam follows the worldline between $(0,0)$ and $(\Delta\hat{t}, 0)$ and diminishes intensity by one half. Here $\Delta\hat{t} = 1.8 \times 10^{-8}$ seconds.

Transforming to the lab frame, the beam starts at $(0,0)$ and reaches

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma(v) \begin{pmatrix} 1 & \frac{v}{c} \\ \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} c\Delta\hat{t} \\ 0 \end{pmatrix} = \gamma(v)\Delta\hat{t} \begin{pmatrix} c \\ v \end{pmatrix} \tag{22}$$

when the beam has diminished in intensity by one half. Note that the x coordinate is

$$\gamma(v)\Delta\hat{t}v = (7.08) \cdot 1.8 \times 10^{-8} \cdot (.99c) = 38m \tag{23}$$