**1.** [Polarization Identity] Let $x \cdot y$ denote the standard dot product in $\mathbb{R}^n$ where $n = 2$ or $n = 3$. and let $|x|$ denote the length of $x$, so

$$|x| = \sqrt{x \cdot x}. \tag{1}$$

Show that for all $x, y \in \mathbb{R}^n$,

$$x \cdot y = \frac{1}{2}\left[|x+y|^2 - |x|^2 - |y|^2\right]. \tag{2}$$

This formula is known as the polarization identity, and it shows that if you can compute lengths, then you can also compute the dot product.

**2.** [Symmetric bilinear maps are determined by their action on the diagonal] A map $B : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is called symmetric if $B(x, y) = B(y, x)$ for all $x, y \in \mathbb{R}^3$. We say $B$ is linear in its first argument if $B(x+z, y) = B(x, y)+B(z, y)$ for all $x, y, z$ and if $B(\lambda x, y) = \lambda B(x, y)$ for all $\lambda \in \mathbb{R}$ and all $x, y \in \mathbb{R}^3$. The definition for $B$ being linear with respect to its second argument is similar, and we say that $B$ is **bilinear** if it is linear with respect to both arguments. Suppose $B$ and $\hat{B}$ are both bilinear and symmetric, and that for all $x \in \mathbb{R}^3$,

$$B(x, x) = \hat{B}(x, x). \tag{3}$$

Show that $B(x, y) = \hat{B}(x, y)$ for all $x, y \in \mathbb{R}^3$.

Hint: Try out that polarization trick.

**3.** In this problem you may assume that every isometry of $\mathbb{R}^3$ that sends the origin to itself is a linear map (i.e. there is a $3 \times 3$ matrix $A$ such that $i(x) = Ax$). Under this assumption (which is true!), show that a map $i : \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry if and only if there is a matrix $A$ such that $A^t A = I$ and a vector $b$ such that

$$i(x) = Ax + b. \tag{4}$$

Hint: Preserving distance is the same as preserving distance squared! And think about polarization.

**4.** Suppose that $A \in O(3)$. Show that $A \in SO(3)$ if and only if for all vectors $x, y, z \in \mathbb{R}^3$ such that $(0, x, y, z)$ is positively oriented, so is $(0, Ax, Ay, Az)$.

Bonus: Show that if for for just one positively oriented $(0, x, y, z)$ we know that $(0, Ax, Ay, Az)$, show that we can still conclude that $A \in SO(3)$.

**5.** SR 4.1 (ii), (iii)

**6.** SR 4.2