Antiderivatives

So far, we've taken a function, and computed its derivative (i.e. the rate of change).

Often in life, we need to go backwards: from a rate of change, compute the original function.

E.g. Given water in a tank, if you knew the rate at which water is draining, you'd like to compute the amount of water in the tank.

Def: An <u>antiderivative</u> of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

This is the game: I give you $f(x)$, you find $F(x)$.

There is a catch:

e.g. Find a function $F(x)$ such that

$$F'(x) = 0 \text{ for all } x. \quad (\text{I gave you } 0)$$

Well, $F(x) = 5$ everywhere will do.

But so will $F(x) = 8$. Or $F(x) = \pi$!

$F(x) = C$ for any constant C will work.

Is that all of them? From the Mean Value Theorem:

$\boxed{\begin{array}{l}\text{If } F(x) \text{ is a function defined on an interval} \\ \text{and } F'(x)=0 \text{ for all } x, \text{ then } F \text{ is constant.}\end{array}}$

e.g.

Find a function $F(x)$ with $F'(x) = x^2$.

A little cleverness: $F(x) = \frac{x^3}{3}$ will do.

Are there any others? Sure. $F(x) = \frac{x^3}{3} + e^{\pi}$

$$F(x) = \frac{x^3}{3} + 19$$

In fact, suppose $G(x)$ is a function with $G'(x) = x^2$.

$$\frac{d}{dx}\left(G(x) - \frac{x^3}{3}\right) = G'(x) - x^2 = x^2 - x^2 = 0$$

So $G(x) - \frac{x^3}{3} = C$ for some C

$$G(x) = \frac{x^3}{3} + C$$

Upshot:

- If you can find one antiderivative $F(x)$ of $f(x)$ you can find lots: $F(x)+C$ $C\in\mathbb{R}$.
- If the domain of f is an interval, that's <u>all</u> of them.

e.g. Find all antiderivatives of $\sin(x)$.

By cleverness $\frac{d}{dx}(-\cos(x)) = \sin(x)$.

So all antiderivatives have the form

$F(x) = -\cos(x) + C$

Bad news: taking derivatives is easy.

finding antiderivatives is hard
(or impossible if you ask for too much).
(I can give you one, but you won't like it).

Generally requires cleverness.

Some rules to help you, based on derivative rules:

$$\frac{d}{dx}\left(aF(x)\right) = aF'(x)$$

$$\frac{d}{dx}\left(F(x)+G(x)\right) = F'(x)+G'(x)$$

Read them backwards and you get

Thm: If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ (so $F'(x)=f(x)$ and $G'(x)=g(x)$)

then

- $aF(x)$ is an antiderivative of $af(x)$ $\forall a\in\mathbb{R}$
- $F(x)+G(x)$ — — — — $f(x)+g(x)$.

e.g. Find an antiderivative of $f(x) = x^2 + 7\sin(x)$

antiderivative of x^2: $\frac{x^3}{3}$

antiderivative of $\sin(x)$: $-\cos(x)$

antiderivative: $F(x) = \frac{x^3}{3} - 7\cos(x)$.

e.g. Find all antiderivatives of $x^2 + 7\sin(x)$:

$$F(x) = \frac{x^3}{3} - 7\cos(x) + C$$