L'Hôpital's Rule:

When we started in with limits we motivated them by observing we need to deal with $\frac{0}{0}$ to cope with instantaneous rates of change. And we built derivative technology to avoid dealing with these limits directly

But we get a sort of payback at this point: we can use derivatives to compute $\frac{0}{0}$ limits:

Let's see it in action:

$$\lim_{x \to 0} \frac{\sin(x)}{x} \qquad \lim_{x\to 0} \sin(x) = \sin(0) = 0 \qquad \frac{0}{0}$$

$$\lim_{x \to 0} x = 0$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}\sin(x)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \frac{\cos(0)}{1} = 1.$$

It's gonna be that easy.

L'Hôpital's Rule (Basic Version)

Suppose f, g are differentiable at a and $f(a) = g(a) = 0$, and $g'(x) \neq 0$ near a, except possibly at a.

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

so long as the latter limit exists or is $\pm\infty$.

e.g. $$\lim_{x \to 2\pi} \frac{\cos(x) - 1}{x - 2\pi}$$ $\cos(2\pi) - 1 = 0$ ✓

$\cancel{2\pi} - 2\pi = 0$ ✓

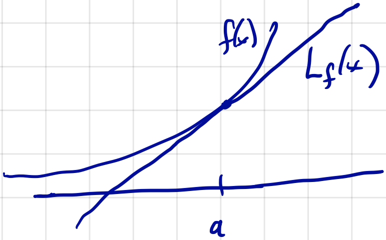
$$\lim_{x \to 2\pi} \frac{\cos(x) - 1}{x - 2\pi} = \lim_{x \to 2\pi} \frac{-\sin(x)}{1} = -\sin(2\pi) = -0 = 0.$$

You absolutely, positively must verify the hypotheses: you're looking for 0/0.

$$\lim_{x \to 2\pi} \frac{\cos(x)}{x} = \frac{\cos(2\pi)}{2\pi} = \frac{1}{2\pi}$$

not $$\frac{-\sin(2\pi)}{1} = 0$$

Why does this work? Sketch:



$f(x) \approx L_f(x)$ for x near a.

And if $f(a)=0$

$L_f(x) = f(a) + f'(a)(x-a)$ $\qquad f(a)=0$

$L_f(x) = f'(a)(x-a)$

$L_g(x) = g'(a)(x-a)$

x near a

$$\frac{f(x)}{g(x)} \approx \frac{L_f(x)}{L_g(x)} = \frac{f'(a)(x-a)}{g'(a)(x-a)}$$

So $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} \approx \lim_{x\to a} \left[\frac{f'(a)(x-a)}{g'(a)(x-a)}\right] = \frac{f'(a)}{g'(a)}$

There are variations on the rule:

0) Works for $\frac{\infty}{\infty}$, not just $\frac{0}{0}$.

1) It applies to limits at ∞:

$$\lim_{x\to\infty} f(x) = 0 \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to\infty} g(x) = 0$$

e.g: $$\lim_{x\to\infty} \frac{\overset{\to\infty}{x}}{\underset{\to\infty}{e^x}} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} \approx 0$$

2) You can use it if the result is $\pm\infty$, and for one-sided limits

e.g $$\lim_{x\to 0^+} \frac{x}{\cos(x)-1} = \lim_{x\to 0^+} \frac{1}{-\sin(x)} = \frac{1}{0^-} = -\infty$$

3) You can use it for indeterminate forms $0 \cdot \infty$ if you can massage into $\frac{0}{0}$ or $\frac{\infty}{\infty}$

e.g. $\lim_{x \to \infty} x^3 e^{-x}$ $x^3 \to \infty$ $e^{-x} \to 0$ $\infty \cdot 0 = ?$

$$= \lim_{x\to\infty} \frac{x^3}{e^x} \underset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \underbrace{\frac{3x^2}{e^x}}_{\frac{\infty}{\infty}} \underset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \underbrace{\frac{6x}{e^x}}_{\frac{\infty}{\infty}} \underset{\text{l'Hôpital}}{=} \lim_{x\to\infty} \frac{6}{e^x} = \frac{6}{\infty} = 0.$$

4) And even 1^∞, 0^0, ∞^0 with more massaging using $\ln$, e^x:

e.g. $\lim\limits_{x \to 0} x^x$

a) apply ln, compute limit

$0 \cdot \infty$ $\frac{\infty}{\infty}$ $\frac{-\infty}{\infty}$ ✓

$$\lim_{x \to 0} \ln(x^x) = \lim_{x \to 0} x \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} \frac{-x^2}{x}$$

$$= 0$$

b) apply e^x

$$\lim_{x \to 0} x^x = \lim_{x \to 0} e^{\ln(x^x)} = e^0 = 1$$

And even $\infty - \infty$ sometimes

$$\lim_{x \to 1^+} \frac{1}{\ln(x)} - \frac{1}{x-1} \qquad \infty - \infty$$

$$= \lim_{x \to 1^+} \frac{(x-1) - \ln(x)}{(x-1)\ln(x)} \qquad \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\ln(x) + \frac{x-1}{x}} \quad \frac{0}{0} \quad \stackrel{H}{=} \lim_{x \to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \lim_{x \to 1^+} \frac{1}{x+1} = \frac{1}{2}$$