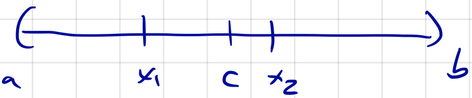
Last class: MVT

$$\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for some } c \text{ in } (a,b)$$

Legalese: f continuous on $[a,b]$
f diff on (a,b).

We used it to show that if

$f'(x)=0$ on (a,b) then f is constant.



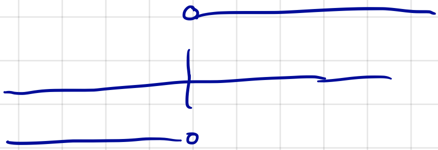
$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) = 0 \Rightarrow f(x_2)=f(x_1)$$

for all x_1, x_2

Easy: if $f(x)=C$ then $f'(x)=0$

Hard: if $f'(x)=0$ then $f(x)=C$

(and this needs the domain to be an interval)



$f'(x) = 0$ on its domain,

f is not constant.

Cor: If f and g are diff on (a,b) and $f'(x) = g'(x)$ for all x in (a,b) then there is a C

$$f(x) = g(x) + C.$$

Pf: Consider $h(x) = f(x) - g(x)$.

Then $h'(x) = f'(x) - g'(x) = 0$.

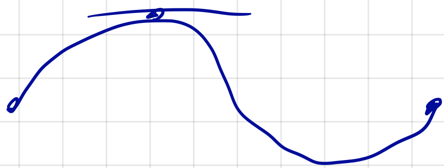
So $h = C$. So $f(x) = g(x) + C$.

Mantra: if two functions have the same derivative (and their domain is an interval) then they differ at most by a constant.

Rolle's Lemma:

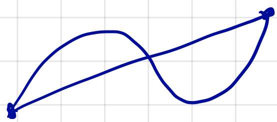
If f is cts on $[a,b]$, and diff on (a,b),
and if $f(a) = f(b)$ there exists c in (a,b)
where $f'(c) = 0$

Pf: By the extreme value theorem, f attains
a min at some c_1 and a max at some c_2.
If c_1 or c_2 is in (a,b), $f'(c_i) = 0$.
Otherwise both are at the ends and
the min and max are the same. So the
function is constant and $f' \equiv 0$.



MVT.

Consider $h(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}\right](x-a)$



$h(a) = f(a)$

$h(b) = f(b) - \frac{(f(b)-f(a))}{b-a} b-a$

$= f(a).$

So by Rolle's Lemma, $h'(c)=0$ for some c.

But $h'(x) = f'(x) - \left[\frac{f(b)-f(a)}{b-a}\right].$

So $f'(c) = \frac{f(b)-f(a)}{b-a}.$

A consequence of the MVT:

If $f'(x) > 0$ on $(a,b]$

then f is strictly increasing on (a,b).

i.e. if $x_1 < x_2$ in (a,b)

$$f(x_1) < f(x_2).$$

$$f(x_2) = f(x_1) + \underbrace{f'(c)}_{>0} \underbrace{(x_2 - x_2)}_{>0}$$

Similarly, if $f'(x) < 0$ on (a,b),

f is strictly decreasing on (a,b)

This is known as the inc/dec test

Take a moment:

$$f(x) = \frac{2}{3}x^3 + x^2 - 12x + 7$$

On what intervals is it increasing/decreasing?

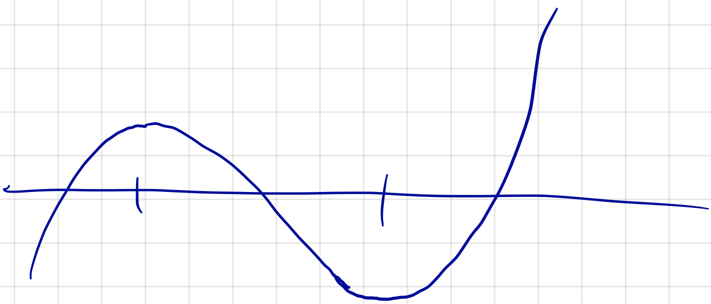
$$f'(x) = 2x^2 + 2x - 12$$
$$= 2\left[x^2 + x - 6\right]$$
$$= 2(x+3)(x-2)$$

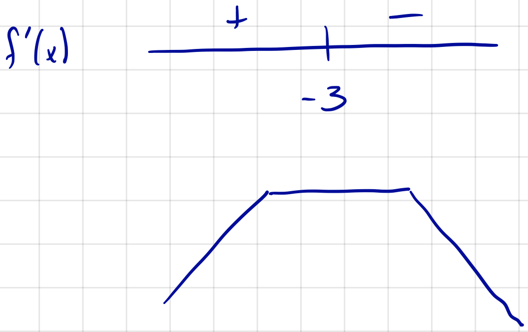
$x+3$: $-$ | -3 | $+$

$-$ | 2 | $+$

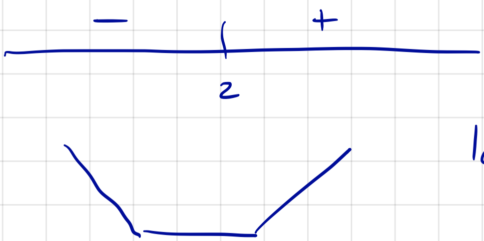
$f'(x)$: $+$ | $-$ | $+$

inc. | dec. | inc.





local max!



local min!

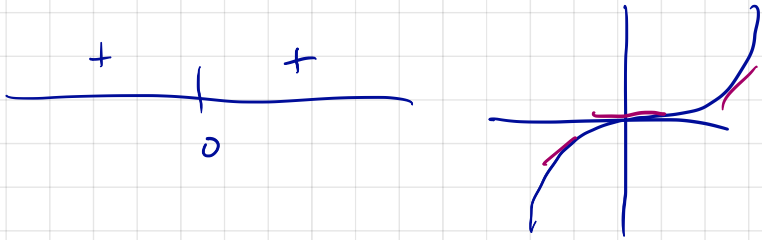
First derivative test:

Suppose $f(x)$ is diff on an open interval containing c, $f'(c)=0$, and $f'(x)$ changes from positive to negative as x increases through c.
Then f has a local max at c.

If f changes from neg to pos, f has a local min.

If f does not change sign then c is neither a local min nor a local max.

e.g. $f(x) = x^3 \quad f'(x) = x^2$



There is a related test.

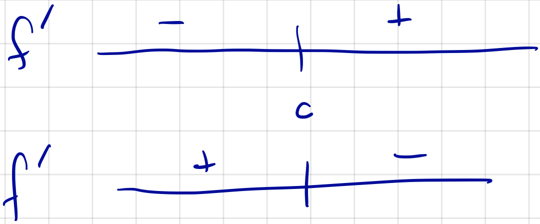
Second Deriv Test

Suppose f is defined on an open interval, $f''(x)$ is continuous on that interval and

$$f'(c) = 0$$
$$f''(c) > 0.$$

$f''(c) > 0$ + cts $\Rightarrow$ $f''(x) > 0$ near c.

$\Rightarrow$ $f'(x)$ is increasing.



which?

First one, so by first derivative test, f attains a local <u>min</u> at c.

Similarly, if $f'(c) = 0$, $f''(c) < 0 \Rightarrow$ local max at c.

Caution: $f'(c), f''(c) = 0 \Rightarrow$ inconclusive.

e.g. x^4, $-x^4$, x^3

e.g. $f(x) = xe^x$

$$f'(x) = e^x + xe^x = (1+x)e^x$$

$$f'(x) = 0 \text{ at } x = -1.$$

Local min or max?

$(1+x)$: $-$ | $+$ at -1

$e^x(1+x)$: $-$ | $+$ at -1

local min!

$$f''(x) = e^x + (1+x)e^x$$

$$= (2+x)e^x$$

$f''(-1) = 1 \cdot e^{-1} > 0 \Rightarrow$ local min!