Optimization:

We can use calculus in some settings to find "the best" option among many possibilities. Often best is formulated as biggest.

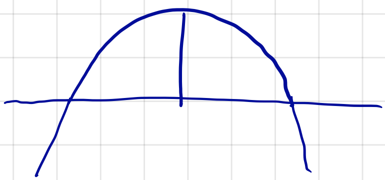
Suppose f is a function with domain $\mathbb{R}$, say.

If c is a point in $\mathbb{R}$ and if

$$f(c) \geq f(x) \text{ for all } x \in \mathbb{R}$$ we

say $f(c)$ is an <u>absolute maximum value</u> of f.

e.g.



$f(x) = 1 - x^2$ on $(-1, 1)$

$f(0) = 1$ is a maximum value for $f(x)$

If $f(x) = 5$ for all x, does f have a max value?

Where is the max achieved?

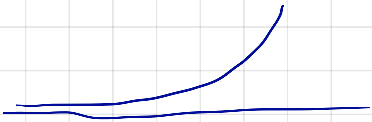
We have a similar concept for minimums.

$f(c)$ is an <u>absolute</u> <u>minimum</u> value, if $f(c) \leq f(x)$ for all $x \in \mathbb{R}$

Note: f from the previous example does not have an absolute minimum.

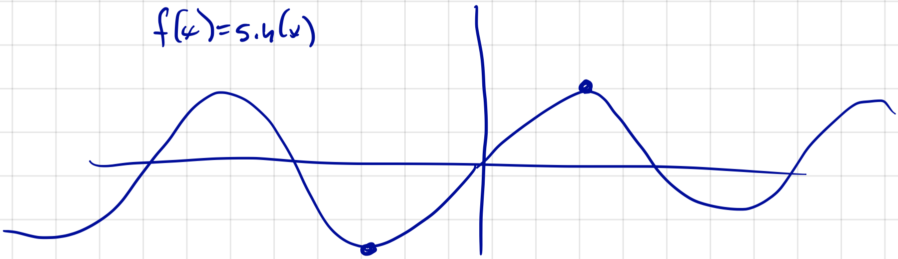
Functions need not have any absolute maximums or minimums.

e.g. $f(x) = e^x$



And they can have multiple:

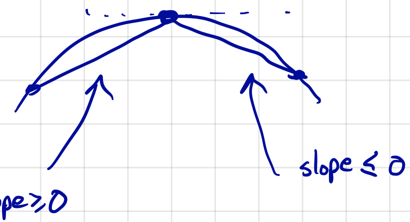
$f(x) = \sin(x)$



f attains its max at $c = \frac{\pi}{2} + k2\pi$

1 is a max value
-1 is a min value

What does calculus have to do with this?



intuitively, slope of tangent line there is 0.

Fact: If the domain of f contains an interval around c, and if f attains a max or min at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Need all of these:

<u>Both sides</u>

$f(x) = x$ on $[-1,1]$.

Attains a max at $x=1$

$f'(1) = 1 \neq 0$.

<u>Derivative exists</u>

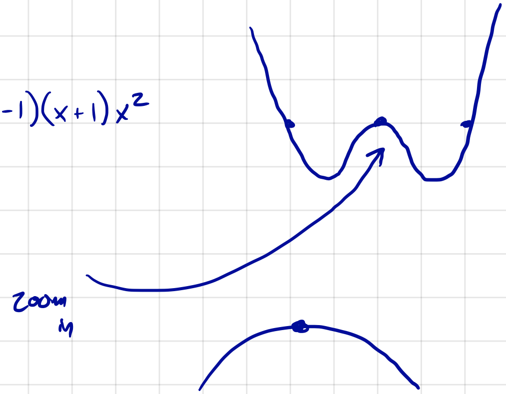
$f(x) = |x|$ on $[-1,1]$

Attains a min of 0 at $x=0$.

But $f'(0)$ does not exist.

Here's a more nuanced version of a maximum value:

$f(x) = (x-1)(x+1)x^2$



If x is "near" $c = 0$, $f(x) \leq f(c)$.

We say $f(c)$ is a <u>local maximum value</u> of f if $f(x) \leq f(c)$ for x near c.

(And a similar definition for a local min.)

Typically, we don't want local min/maxs but we need to be aware of them.

Fermat's Theorem

Fact: If the domain of f contains an interval around c,
and if f attains a local max or min at c,
and if $f'(c)$ exists, then $f'(c) = 0$.

This gives you a way to look for maxima:

1) Spots where $f'(x) = 0$

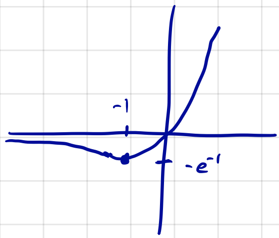
2) Spots where $f'(x)$ does not exist

3) spots x where f is not defined on both sides of x.

It's gotta be one of these, if anywhere.

Def: we say c is a <u>critical point</u> for f
if $f'(c) = 0$ or $f'(c)$ DNE.

e.g. $f(x) = xe^x$



clearly doesn't have an absolute max.

Could have an absolute min. $f'(x) = e^x + xe^x = (1+x)e^x$

$f'(x) = 0$ at $x = -1$.

Indeed $f(-1) = -e^{-1}$ is a minimum value for f.

Now we have seen that functions need not have mins/maxes.

But: If f is continuous on $[a,b]$, a closed, bounded interval, then f achieves both a minimum and a maximum.

(Extreme Value Theorem)

e.g. Find maximum and minimum values of

$f(t) = t - t^{1/3}$ on $[-1, 4]$

$f'(t) = 1 - \frac{1}{3} t^{-2/3}$

$f'(t) = 0 \qquad 3 = t^{-2/3}$

$t = 3^{-3/2}$

derivative defined everywhere

Three candidates: $x = -1$
$x = 3^{-3/2}$
$x = 4$

$f(-1) = -1 - (-1) = 0$
$f(4) = 4 - 4^{1/3} = 2.41\ldots$
$f(c) = -0.3849\ldots$

$$f(x) = x + \frac{1}{x} \quad \text{on } [1/5, 4]$$

$$f'(x) = 1 - \frac{1}{x^2}$$

$f'(x) = 0$ at 1

candidates: $x = 1/5, 0, 4$

$$f(1/5) = 5 + \frac{1}{5} = 5.2$$

$$f(4) = 4 + \frac{1}{4} = 4.25$$

$$f(1) = 1 + 1 = 2$$

So f attains a minimum of 2 at $x = 1$
max of 5.2 at $x = 0.2$

$f(x) = x^{2/3}$ on $[-8, 8]$

$f'(x) = \frac{2}{3} x^{-2/3} \neq 0$

$f'(0)$ DNE, so this is a crit point.

$$\left.\begin{aligned} f(-8) &= 4 \\ f(8) &= 4 \end{aligned}\right] \text{max}$$

$$f(0) = 0 \quad \left.\right] \text{min.}$$