$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{x(x+h)} \frac{1}{h} = \frac{-1}{x^2}$$

---

$f(x) = 3^x$

| | |
|---|---|
| $h = 0.1$ | 1.1612 |
| $h = 0.01$ | 1.047 |
| 0.001 | 1.099 |
| | 1.09867 |
| 0.0001 | 1.09861 |

$f'(1) \approx 1.099$

$f'(x) \approx (1.099)\,3^x$

---

$$\frac{d}{dx}\sqrt{x} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We saw $\frac{d}{dx}\left[f(x)\,g(x)\right] \neq f'(x)\,g'(x)$

$$\frac{d}{dx}\, x \cdot x = \frac{d}{dx} x^2 = 2x$$

but $\left(\frac{d}{dx} x\right)\left(\frac{d}{dx} x\right) = 1 \cdot 1 = 1 \neq 2x.$

Instead:

$$f(x+h)\,g(x+h) - f(x)\,g(x) =$$

$$f(x+h)\,g(x+h) - f(x+h)\,g(x) + f(x+h)\,g(x) - f(x)\,g(x)$$

$$\lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} = \lim_{h \to 0} f(x+h)\left(\frac{g(x+h) - g(x)}{h}\right)$$

$$+ \lim_{h \to 0} \frac{\left(f(x+h) - f(x)\right)}{h}\, g(x)$$

continuity

$$= f(x)\,g'(x) + f'(x)\,g(x)$$

Product rule:

$$\frac{d}{dx} f(x)g(x) = \left[\frac{d}{dx} f(x)\right] g(x) + f(x)\left[\frac{d}{dx} g(x)\right]$$

Does this work?

$$\frac{d}{dx} x^2 = \left(\frac{d}{dx} x\right) x + x \frac{d}{dx} x$$

$$= 1 \cdot x + x \cdot 1 = 2x \checkmark$$

$$\frac{d}{dx} x^3 = \frac{d}{dx} x\, x^2 = \left(\frac{d}{dx} x\right) x^2 + x \frac{d}{dx} x^2$$

$$= x^2 + x \cdot 2x = 3x^2.$$

Fun!

$$\frac{d}{dx} x^{n+1} = \frac{d}{dx} x x^n = x^n + x \frac{d}{dx} x^n$$

If for some $n$ $\frac{d}{dx} x^n = n x^{n-1}$

Then $\frac{d}{dx} x^{n+1} = x^n + x n x^{n-1}$

$$= x^n + n x^n$$

$$= (n+1) x^n$$

So $\frac{d}{dx} x^3 = 3x^2$

$\frac{d}{dx} x^4 = 4x^3$

$\frac{d}{dx} x^5 = 5x^4$

etc!

e.g.

$$\frac{d}{dx}\left(x^3-2x^2+1\right)e^x$$

$$= \left(3x^2-4x\right)e^x + \left(x^3-2x^2+1\right)e^x$$

$$= \left(x^3+x^2-4x+1\right)e^x$$

Worksheet: $\frac{d}{dx}x^{-1} = \frac{-1}{x^2} = (-1)x^{-2}$ !

Inverse rule

$$\frac{d}{dx}\frac{1}{f(x)} = \lim_{h\to 0}\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

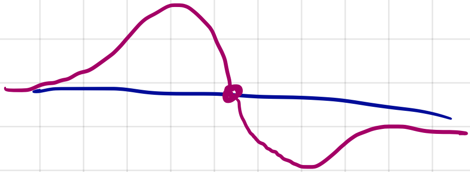
$$= \lim_{h\to 0}\frac{f(x) - f(x+h)}{h\, f(x)\, f(x+h)}$$

$$= \frac{1}{f(x)} \cdot \lim_{h\to 0}\frac{1}{f(x+h)} \cdot (-1)\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \frac{-1}{f(x)^2}\, f'(x)$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{f(x)^2}$$

e.g. $$\frac{d}{dx}\frac{1}{1+x^2} = \frac{-2x}{(1+x^2)^2}$$



Quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)}{g(x)} + f(x)\frac{d}{dx}\left(\frac{1}{g(x)}\right)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g(x)^2}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\frac{e^x}{1-e^x} = \frac{e^x(1-e^x) - e^x(-e^x)}{(1-e^x)^2}$$

$$= \frac{e^x}{(1-e^x)^2}$$