Derivatives + Rates of Change.

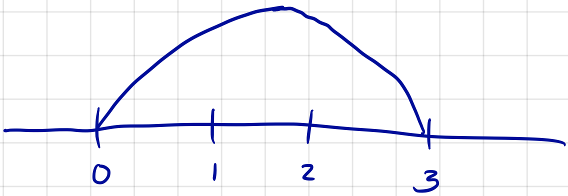
Suppose a ball is tossed in the air

$y(t)$ is height of ball, in m. t in s.

$y(0) = 0$, say

$y(t) = 15t - 5t^2$

$y(0) = 0 \qquad y(3) = 15 \cdot 3 - 5 \cdot 3 \cdot 3 = 0$



$y(1) = 10$

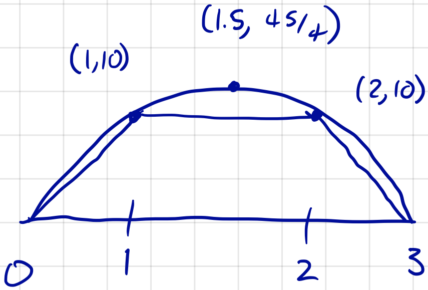
$y(2) = 10 \qquad 30 - 20 \checkmark$

Average rates of change of height:

$[0,1] \qquad \dfrac{y(1) - y(0)}{1 - 0} = 10 \text{ m/s}$

$[1,2] \qquad \dfrac{y(2) - y(1)}{2 - 1} = \dfrac{10 - 10}{1} = 0 \text{ m/s}$

$[2,3] \qquad \dfrac{y(3) - y(2)}{3 - 2} = \dfrac{0 - 10}{1} = -10 \text{ m/s}.$



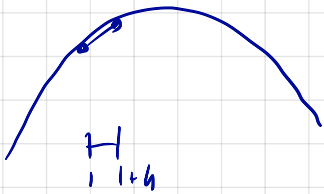
The slopes of the three secant lines equal the average rates of change over the three time intervals.

What is the instantaneous rate of change in height (velocity) at $t=1$?

We can estimate: 10 or 0, or better yet $\frac{10+0}{2}$.

To formalize:

$$\frac{y(1+h)-y(1)}{h}$$



interpret as either

a) avg rate of change over $[1, 1+h]$

b) slope of secant line.

As we take the limit as $h \to 0$
we pick up the instantaneous rate of change of height
(AKA velocity)

or the slope of the tangent line.

$15t - 5t^2$

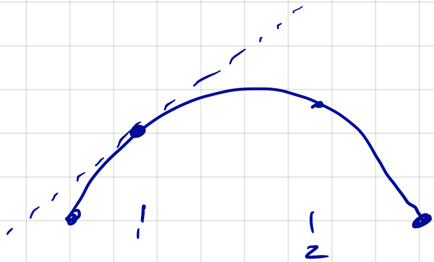
Let's do it:

$$y(1+h) = 15(1+h) - 5(1+h)^2$$
$$= 15 + 15h - 5(1+2h+h^2)$$
$$= 10 + 15h - 10h - 5h^2$$
$$= 10 + 5h - 5h^2$$

$$y(1) = 15 \cdot 1 - 5 \cdot 1^2 = 10$$

$$\frac{y(1+h) - y(1)}{h} = \frac{5h - 5h^2}{h}$$ but you can't set $h = 0$.

But $\lim\limits_{h \to 0} \frac{5h - 5h^2}{h} = \lim\limits_{h \to 0} 5 - 5h = \boxed{5}$



The velocity at $t=1$ is 5 m/s.

The slope of the tangent line is 5.

What's the equation of the tangent line?

Forms $\quad y = mx + b \quad$ blech.

$$y - y_1 = m(x - x_1)$$

point slope!
If you know m, y_1, x_1
good to go.

$x_1 = 1, \quad y_1 = 10$

$$y - 10 = 5(x - 1)$$

$$y = 5x + 5$$

slope 5? ✓
(1, 10) ? ✓

This limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ is known as the

<u>derivative</u> of f at a

and has the notation $f'(a)$.

Alternatively $b = a+h$ $(h = b-a)$

as $h \to 0$, $b \to a$.

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

You need to know both expressions.

What does the derivative tell you?

Think of population at time t

$$\frac{P(b)-P(a)}{b-a} \qquad \frac{\Delta P}{\Delta t}$$

If b is close to a, $\qquad \frac{\Delta P}{\Delta t} \approx P'(a)$

$$\boxed{\Delta P \approx P'(a)\,\Delta t}$$

If you change time by a little bit Δt,
the population will approximately
change by $P'(a)\,\Delta t$

E.g. if $P'(3) = 122 \frac{\text{caribou}}{\text{year}}$ (t in years)

then over 6 months, pop will change by

$$\Delta P = P'(3)\,\Delta t = 122 \frac{\text{caribou}}{\text{year}} \cdot \frac{1}{2} \text{ year} = 61 \text{ caribou per year}$$

Now you:

The radius of a tree is growing.

$r(t) = 10\sqrt{t}$ r in cm, t in years.

a) Find the average rate of change of growth from $t=1$ to $t=2$.

b) Find the instant rate of change of growth at $t=1$.

c) I promise you

$r(4) = 20$

$r'(4) = 5/2$

$10 \frac{1}{2} \frac{1}{\sqrt{t}}$

use these fact alone to approximate r at 4 years and 1 month.

$20 + \frac{5}{2} \cdot \frac{1}{12}$

Then compare your est.