The goal of this worksheet is for you to work through some applications of the Fundamental Theorem of Linear Algebra:

$$\begin{aligned} C(A) &= N(A^T)^\perp \\ C(A^T) &= N(A)^\perp \end{aligned}.$$

For the first set of exercises consider

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -3 \\ 3 & 0 \\ 2 & 1 \end{bmatrix}.$$

Since A is tall and thin, we know that we cannot always solve $A\mathbf{x} = \mathbf{b}$. If we perform elimination, there will be at least two zero rows, and hence there will be at least two conditions on $\mathbf{b}$ needed to ensure solvablility. We would like to identify these conditions.

1. Consider the vector $\mathbf{w} = (-4, -3, 1, -1)$. Show that $A^T\mathbf{w} = \mathbf{0}$.

2. Use this vector to show that for $\mathbf{b} = (-1, 1, 1, 1)$ there is no solution of

$$A\mathbf{x} = \mathbf{b}.$$

Hint: Since $A^T\mathbf{w} = \mathbf{0}$, $\mathbf{w}^T A = \mathbf{0}^T$. Now try to combine this equation with $A\mathbf{x} = \mathbf{b}$.

3. You just saw an example of a general principle. If $A\mathbf{x} = \mathbf{b}$ has a solution, then $\mathbf{w} \cdot \mathbf{b}$ for every vector $\mathbf{w}$ such that $A^T\mathbf{w} = \mathbf{0}$. That is, $C(A) \perp N(A^T)$. We know something better: $C(A) = N(A^T)^\perp$. So $A\mathbf{x} = \mathbf{b}$ will have a solution if and only if $\mathbf{b}$ is in $C(A)$, which happens if and only if $\mathbf{b}$ is perpendicular to every vector in $N(A^T)$. And to detect whether $\mathbf{b}$ is perpendicular to every vector in $N(A^T)$ you only need to show that $\mathbf{b}$ is perpendicular to every vector in a basis for $N(A^T)$. So now the search is on! We need to find a basis for $N(A^T)$. Start by performing elimination on A^T to arrive at row echelon form U. Don't go all the way to R yet! At this point, determine the dimension of $N(A^T)$.

4. Now keep going to reduced row echelon form R. At this point, find a basis for $N(A^T)$.

5. For each vector $\mathbf{w}$ in your basis, double check and verify $A^T\mathbf{w} = \mathbf{0}$.

6. Is there a solution of $A\mathbf{x} = (1, -3, -9, -4)$?

7. Is there a solution of $A\mathbf{x} = (-1, -3, -9, -4)$?

Now suppose we have the vectors $\mathbf{v}_1 = (1, -1, 1)$ and $\mathbf{v}_2 = (1, 1, 2)$. We would like to find a basis for the orthogonal complement of $V = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.

8. Before finding this orthogonal complement, let's get to know V a bit better. First, show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

9. What is the dimension of V? Why?

10. What is the dimension of $V^{\perp}$? Why?

11. To find a basis $V^{\perp}$, the strategy is to construct a matrix A such that $C(A) = V$. Then use $V^{\perp} = C(A)^{\perp} = N(A^T)$. So, write down a matrix A such that $C(A) = V$.

12. Now find a basis for $N(A^T)$, which is your desired basis for $V^{\perp}$. Elimination strikes again!

13. Does your basis have the expected number of elements? Compare with your answer to problem 10.

14. Verify each element of this basis is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$.

15. I bet you have a different, favorite way of finding a vector that is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ that you learned before you took linear algebra. This technique is special to three dimensions, and only works there. Still, if you happen to be working in three dimensions (like we are in this easy example), it's the way to go. Go ahead and use this method to find a vector that is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$