Let

$$A = \begin{bmatrix} 1 & -1 & 5 & 2 & -1 & 10 \\ 2 & -1 & 7 & 4 & 0 & 20 \\ 1 & -1 & 5 & 3 & 0 & 13 \\ 0 & -1 & 3 & 1 & -1 & 3 \end{bmatrix}.$$

1. (4 points) Let's find a single solution of $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (6, 9, 8, 5)$. I'll do some of the work for you. After forming the augmented matrix $[A\ \mathbf{b}]$, I did row operations to arrive at

$$[U\ \mathbf{w}] = \begin{bmatrix} 1 & -1 & 5 & 2 & -1 & 10 & 6 \\ 0 & 1 & -3 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this stage we could find a solution by applying back substitution to three equations involving three unknowns. What are the three equations? **YOU DO NOT NEED TO SOLVE THE EQUATIONS!**

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 6 \\ x_2 \qquad\quad &= -3 \\ x_3 &= 2 \end{aligned}$$

$$[U\ \mathbf{w}] = \begin{bmatrix} 1 & -1 & 5 & 2 & -1 & 10 & 6 \\ 0 & 1 & -3 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. (4 points) Continue doing row operations to bring the system to reduced row echelon form. Then find a single solution $\mathbf{x}_p$ of $A\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & -1 & 5 & 2 & -1 & 10 & 6 \\ 0 & 1 & -3 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 5 & 0 & -3 & 4 & 2 \\ 0 & 1 & -3 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 4 & -1 \\ 0 & 1 & -3 & 0 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = (-1, -3, 0, 2, 0, 0)$$

3. (4 points) For each free column of A, find the corresponding special solution $\mathbf{n}$ of $A\mathbf{n} = 0$.

$$n_1 = \begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad n_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad n_3 = \begin{bmatrix} -6 \\ 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$