F314 Linear Algebra: Quiz 10 November 9, 2017 Name:

1. Let

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{a}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Use the Gram-Schmidt Algorithm to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ that spans the same subspace as $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

You will probably find that computing $\mathbf{q}_1$ and $\mathbf{q}_2$ is pretty easy, and that the real work is in computing $\mathbf{q}_3$.

$$q_1 = a_1 / \|a_1\| = \frac{1}{\sqrt{3}} (1, 1, -1, 0)$$

$$w_2 = a_2 - (a_2 \cdot q_1) q_1$$

$$a_2 \cdot q_1 = 0$$

$$w_2 = a_2$$

$$q_2 = a_2 / \|a_2\| = \frac{1}{\sqrt{3}} (1, -1, 0, 1)$$

$$w_3 = a_3 - (a_3 \cdot q_2) q_2 - (a_3 \cdot q_1) q_1$$

$$a_3 \cdot q_2 = \frac{1}{\sqrt{3}} (1 + 1 + 1) = \frac{3}{\sqrt{3}}$$

$$a_3 \cdot q_1 = \frac{1}{\sqrt{3}} (1 - 1 + 1 + 0) = \frac{1}{\sqrt{3}}$$

$$w_3 = (1, -1, -1, 1) - \frac{3}{3} (1, -1, 0, 1) - \frac{1}{3} (1, 1, -1, 0)$$

$$= (0, 0, -1, 0) - \frac{1}{3} (1, 1, -1, 0)$$

$$= \frac{1}{3} (-1, -1, -2, 0)$$

$$q_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{6}} (-1, -1, -2, 0)$$

2. Use the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ you just discovered to compute the orthogonal projection of $\mathbf{x} = (1, 0, 0, 1)$ onto the space spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$. Start by copying down $\mathbf{q}_1$ and $\mathbf{q}_2$ from the previous page.

$$q_1 = \frac{1}{\sqrt{3}}(1,1,-1,0) \qquad q_2 = \frac{1}{\sqrt{3}}(1,-1,0,1)$$

Method 1)

$$Q = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$y = Q^T x = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$p = Qy = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\frac{1}{3}\begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \end{bmatrix}}$$

Method 2)

$$p = (x \cdot q_1)q_1 - (x \cdot q_2)q_2$$

$$= \frac{1}{\sqrt{3}} q_1 + \frac{2}{\sqrt{3}} q_2$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ -1 \\ -1 \\ 2 \end{bmatrix}$$