1. Suppose a rocket is shot from height $h = 0$ meters at time $t = 0$ seconds. The height of the rocket is known at times $t = 1$, $t = 2$, $t = 3$ and $t = 4$ seconds. These heights are $h_1, \ldots, h_4$, measured in meters.

a) Determine the average velocity of the rocket on the time intervals $[0, 1]$ and $[1, 2]$.

b) Let v_k denote the average vertical velocity on time interval $[k - 1, k]$ and let $\mathbf{v} = (v_1, \ldots, v_4)$. Find a matrix A such that

$$A\mathbf{h} = \mathbf{v}$$

where $\mathbf{h} = (h_1, h_2, h_3, h_4)$. [Note that we are computing velocities from positions; multiplying by matrix A is kind of like taking a derivative.]

c) Suppose you do not know h_2 but you know $h_1 = 100$ m and $v_2 = 300$ m/s. What is h_2?

d) Suppose you do not know h_1 and h_2 but you know $v_1 = 150$ m/s and $v_2 = 300$ m/s. What are the values of h_1 and h_2?

e) Find a matrix S such that if $\mathbf{v}$ is known we can determine $\mathbf{h}$ via

$$\mathbf{h} = S\mathbf{v}.$$

[Notice that multiplying by S converts velocities back into positions. It is a kind of integration operation.]

2. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & 1 & 0 \\ 1 & 3 & -5 \end{bmatrix}.$$

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ denote the columns of A in that order, so $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

a) Show that the columns of A are linearly dependent.

b) Find a nonzero vector $\mathbf{n}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ only if $\mathbf{n} \cdot \mathbf{b} = 0$.

c) Is there a solution if $\mathbf{b} = (-1, 3, 5)$? Why?

d) Find three different solutions of $A\mathbf{x} = \mathbf{0}$.

e) Find three different solutions of $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = (-1, 3, 4)$.

3. Find a 3×3 matrix A such that

$$A\mathbf{x} = \mathbf{b}$$

has a solution if and only if $\mathbf{b}$ is a multiple of $(1, -2, 4)$. That is, for this matrix, a solution exists only if $\mathbf{b}$ lies on a certain line (not just a certain plane!)