1. The matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

has complex eigenvalues.

a) Find both complex eigenvalues, λ_1 and λ_2.

b) Verify that $\overline{\lambda_1} = \lambda_2$. This is a consequence of the fact that complex roots of real polynomials come in complex conjugate pairs.

c) Let λ_1 be the eigenvalue with positive real part. Find an eigenvector $\mathbf{x}_1$ for it.

d) Let $\mathbf{x}_2 = \overline{\mathbf{x}_1}$ and verify that $\mathbf{x}_2$ is an eigenvector for A with eigenvalue $\lambda_2 = \overline{\lambda_1}$.

Remark: This always happens. If λ is a complex eigenvalue of the real matrix A, with eigenvector $\mathbf{x}$, then

$$A\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda x} = \overline{\lambda}\overline{x}.$$

2. Consider the rotation matrix

$$A = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

where c and s are real numbers with $c^2 + s^2 = 1$. We can write $c = \cos\theta$ and $s = \sin\theta$ for some angle θ. Show that A has eigenvalues $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$. Then find an eigenvector for each of these eigenvalues.