Math F314 Final Exam Fall 2012

Name: ______________________

Student Id: ______________________

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Turn off anything that might go beep during the exam.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Good luck!

Problem	Possible	Score
1	14	
2	5	
3	14	
4	17	
5	10	
6	10	
7	10	
8	10	
9	5	
EC	5	
Total	95	

1. (14 points)

Consider the matrix

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -3 & 2 \\ -4 & 15 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

a. (2 points) Find the determinant of A.

b. (7 points) Solve $Ax = b$ where $b = (0, 30, 0)$.

Continued....

Recall that

$$A = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -3 & 2 \\ -4 & 15 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

c. (5 points) Find the second column of A^{-1}. Don't waste time!

2. (5 points)

Consider

$$A = \begin{bmatrix} 1 & 3 & -1 & 3 \\ 2 & -1 & 4 & 0 \\ 6 & -3 & 1 & -1 \\ 2 & 6 & 10 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 0 & 2 \\ -2 & 0 & 1 & 5 \\ -3 & -1 & 4 & -1 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

Write the fourth row of AB as a linear combination of the rows of B.

3. (14 points)

Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

a. (6 points) Determine the eigen**values** of A.

b. (6 points) Let λ_1 be the eigenvalue of A with largest absolute value. Find a corresponding eigenvector $\mathbf{v}_1$.

c. (2 points) What is the value of $A^5\mathbf{v}_1$?

4. (17 points)

Consider

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 6 & 3 & 0 & 10 \\ -4 & -2 & 17 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

a. (4 points) What are the dimensions of the four fundamental subspaces of A?

b. (2 points) Find a basis for the column space of A.

c. (5 points) Find a basis for the null space of A.

Continued....

Recall that

$$A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 6 & 3 & 0 & 10 \\ -4 & -2 & 17 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

d. (2 points) Find a basis for the row space of A.

e. (2 points) Show that $(17, -5, 1)$ is an element of the cokernel of A. Is it a basis?

f. (2 points) Show that there is a solution of $A\mathbf{x} = (1, 3, -2)$ and that there is no solution of $A\mathbf{x} = (1, 3, 2)$. You do **not** need to find the solution in the first case!

5. (10 points)

Determinants!

a. (3 points) Compute the determinant of

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

b. (4 points) Suppose A is a 4×4 matrix and $A^T A = 3A$. What are the possible values of $\det(A)$? Be careful: the matrix is 4×4.

c. (3 points) The three defining characteristics of the determinant were: $|I| = 1$, the alternating property, and the multilinearity property. Using only these properties, show that

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 5 & 3 & 7 \end{vmatrix} = 0.$$

6. (10 points)

Let W be the intersection of the hyperplanes

$$\begin{aligned} w + 2x - 3y + 4z &= 0 \\ -w - x + y + z &= 0. \end{aligned}$$

a. (5 points) Find a basis for W.

b. (5 points) Write down the equations you would solve to find the closest point in W to $(1, 2, 3, 4)$. DO NOT actually do any matrix multiplication, and DO NOT solve the equations.

7. (10 points)

a. (3 points) Let $\mathbf{v}_1 = (2, 1, -2)$, $\mathbf{v}_2 = (2, -2, 1)$, and $\mathbf{v}_3 = (1, 2, 2)$. Show that this is an orthogonal collection of vectors.

b. (7 points) Compute the orthogonal projection of $(1, 3, 1)$ onto $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.

8. (10 points)

a. (3 points) Define what it means for a subset W of $\mathbb{R}^n$ to be a subspace.

b. (4 points) Determine if the collection of vectors (x, y, z) such that $z \neq 3$ is a subspace.

c. (3 points) Determine if the collection of vectors (x, y) such that $xy \geq 0$ is a subspace.

9. (5 points)

Name five conditions that are equivalent to the $n \times n$ matrix A being singular.

10. (Extra Credit: 5 points)

Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Let D be the unit disk in $\mathbb{R}^2$, so D is the set of all vectors in $\mathbb{R}^2$ with length less than or equal to 1. Let $S = \{A\mathbf{x} : \mathbf{x} \in D\}$. That is, if we think of A as defining a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, A takes D to S. What is the area of S?